# A NEW PROOF OF THE FOUR-COLOR THEOREM

Robin Thomas

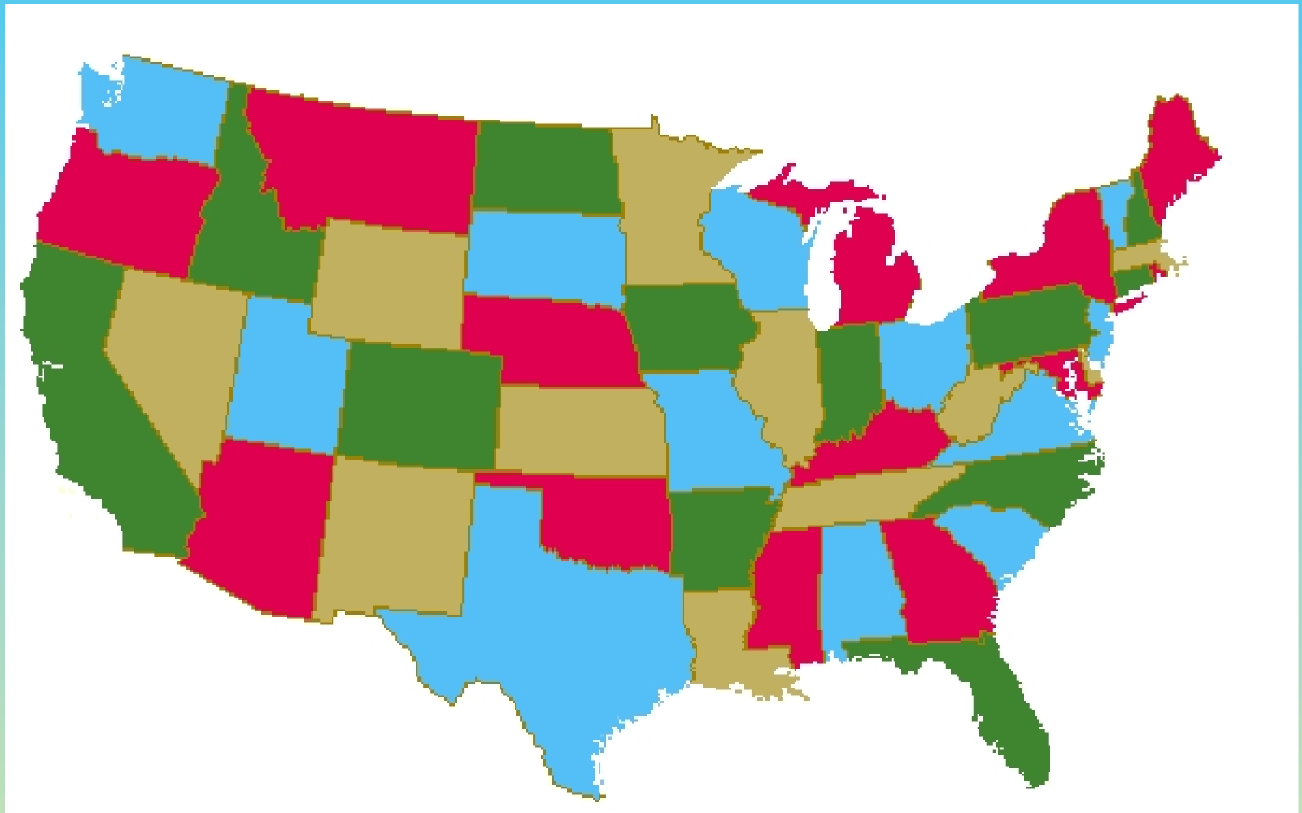
School of Mathematics
Georgia Institute of Technology
www.math.gatech.edu/~thomas

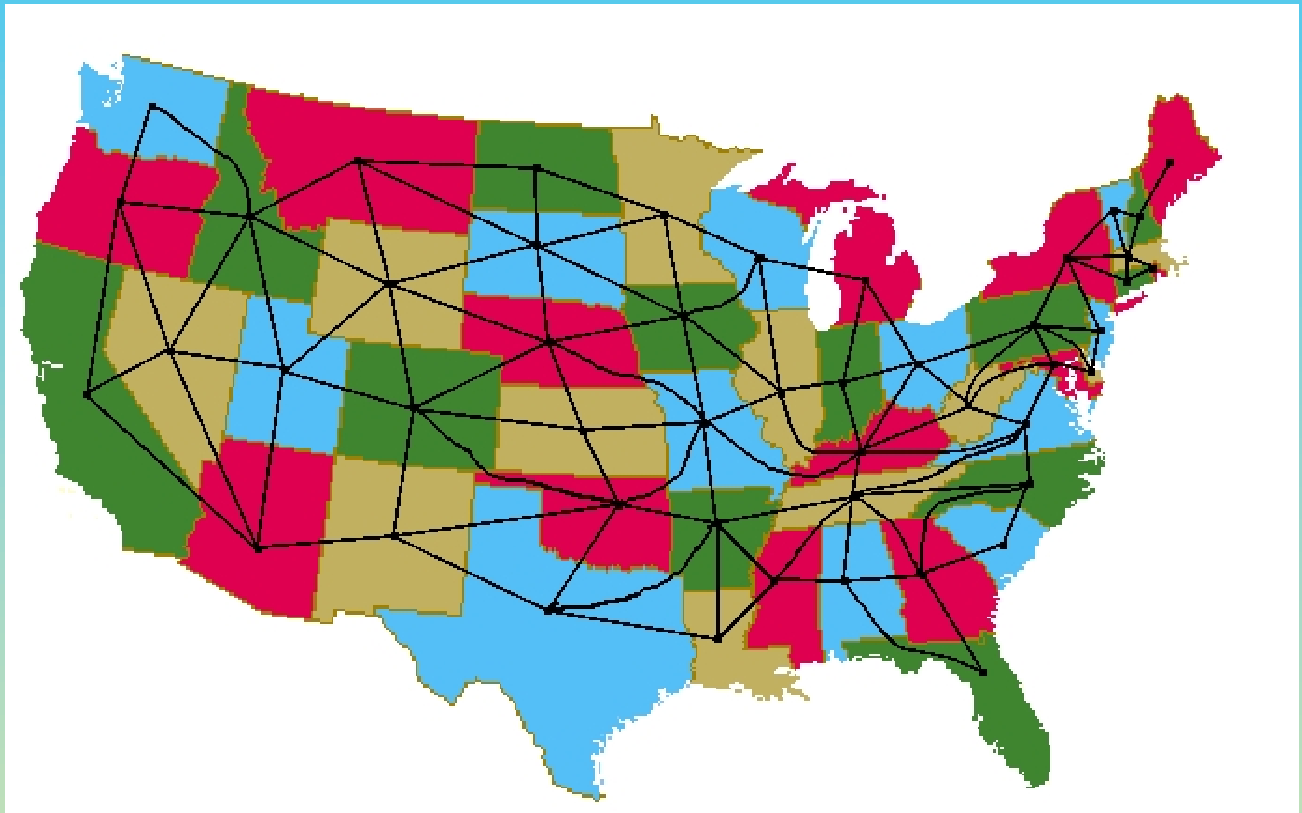
joint work with

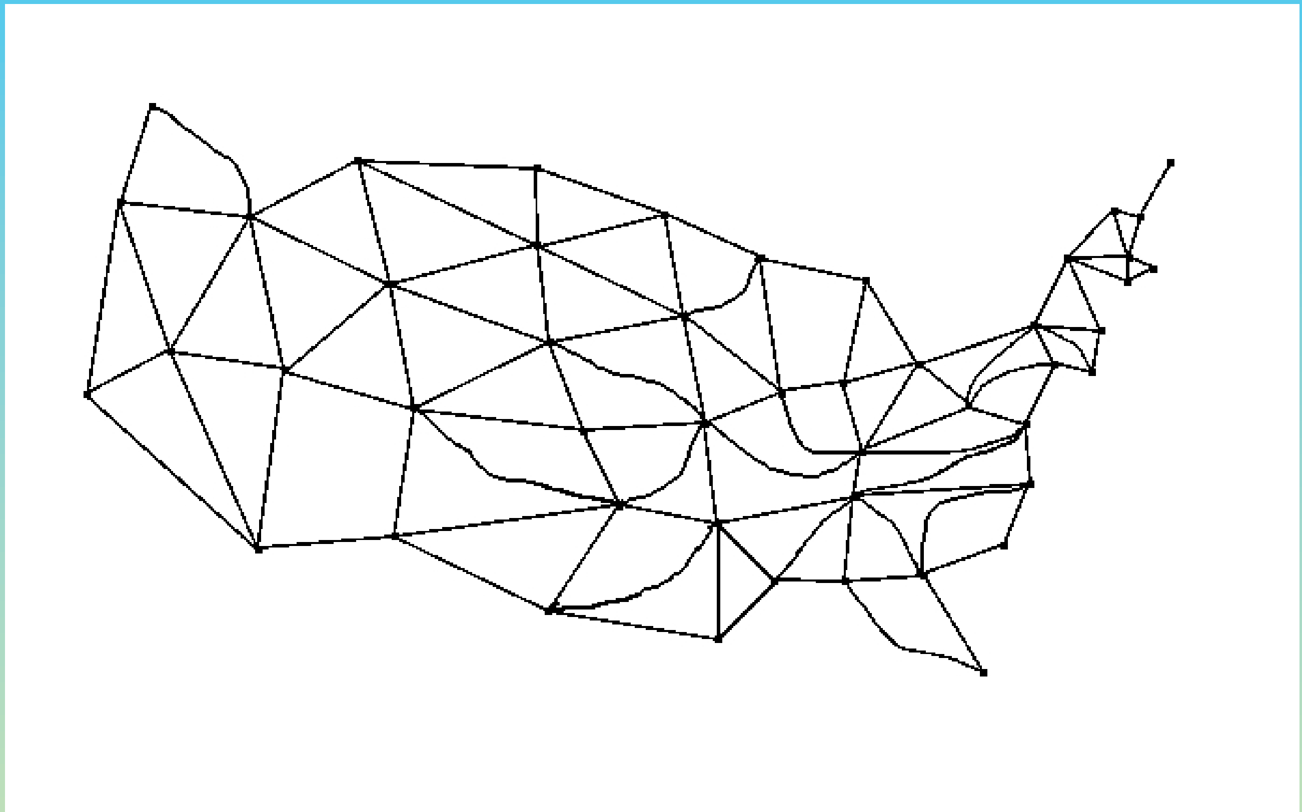
N. Robertson, D. P. Sanders, P. D. Seymour

# OUTLINE

- The statement
- History
- Equivalent formulations
- A proof
- Beyond the Four-Color Theorem







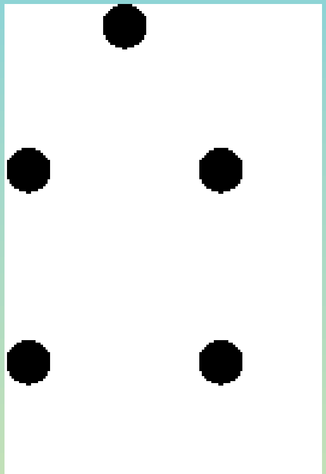
# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

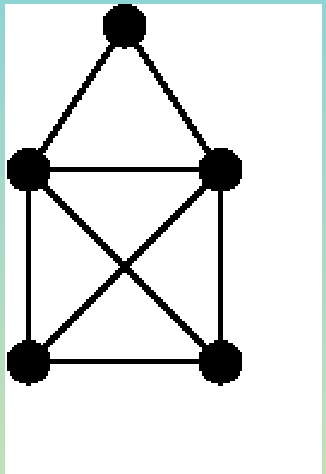
Graphs have vertices



# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

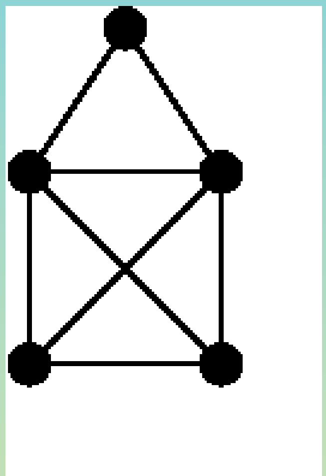
Graphs have vertices and edges.



# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

Graphs have vertices and edges.

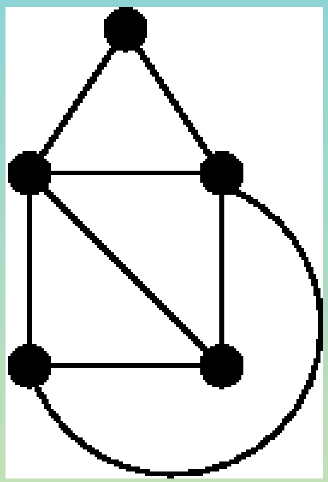


A graph is planar if it can be drawn in the plane without crossings.

# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

Graphs have vertices and edges.

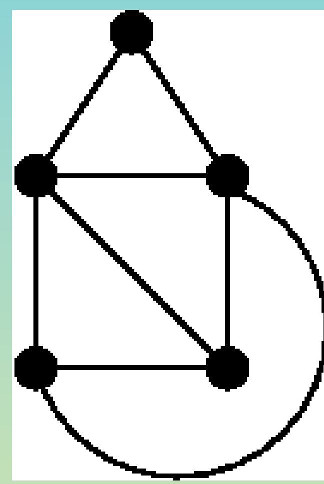


A graph is planar if it can be drawn in the plane without crossings.

# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

Graphs have vertices and edges.

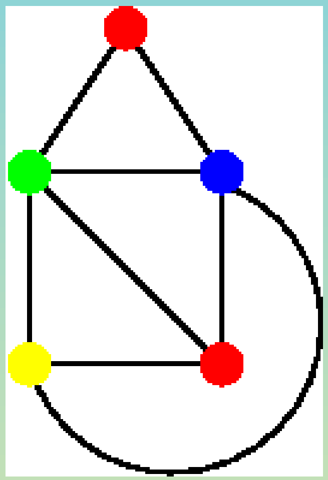


A graph is planar if it can be drawn in the plane without crossings. We want to color so that adjacent vertices receive different colors.

# THE FOUR COLOR THEOREM.

Every planar graph is $4$-colorable.

Graphs have vertices and edges.



A graph is planar if it can be drawn in the plane without crossings. We want to color so that adjacent vertices receive different colors.

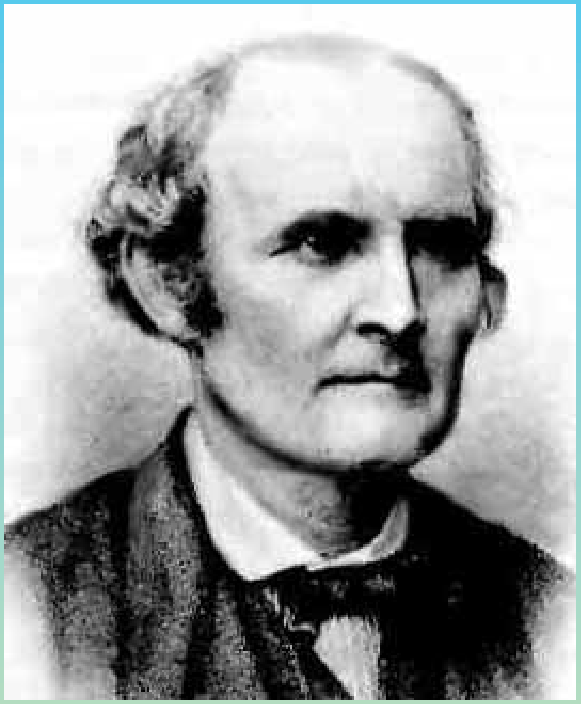
# HISTORY

# Francis Guthrie

- In 1852 colored the map of England with four colors
- asked brother Frederic whether true for all maps
- Frederic communicated the question to de Morgan

# A. de Morgan

- Leading mathematician in London at that time
- Thought seriously about the problem



# Arthur Cayley

- First printed reference in 1878

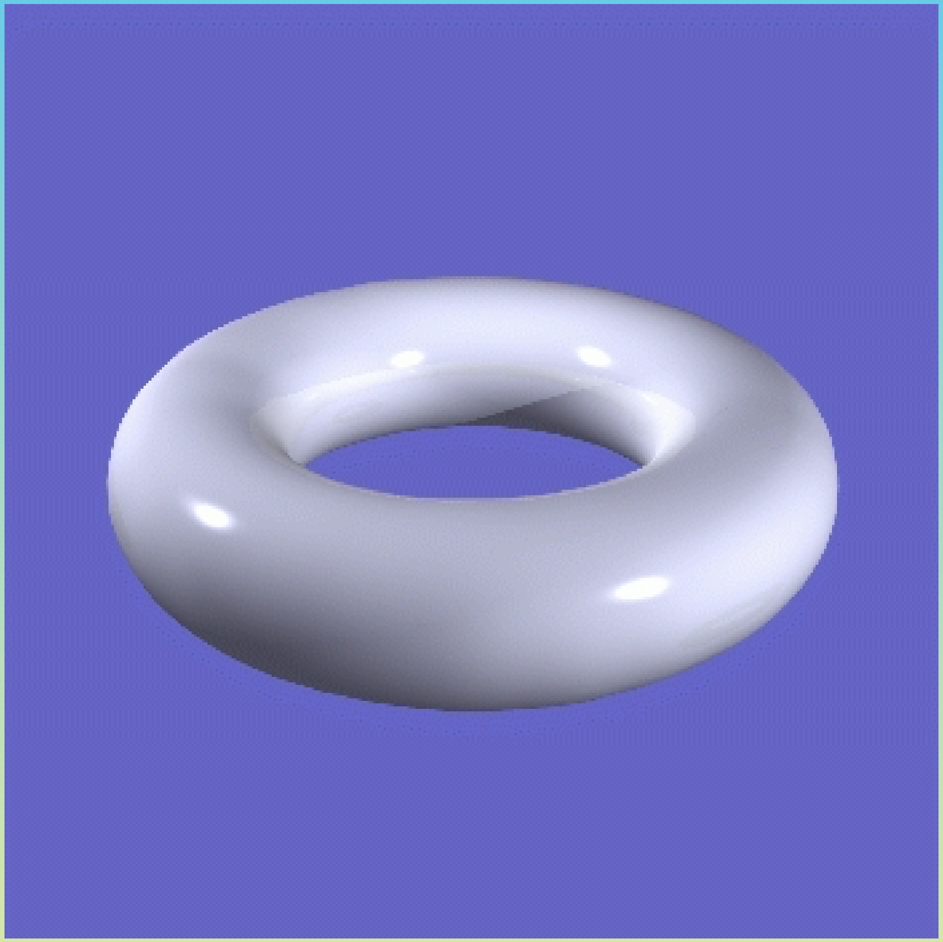
# Alfred Kempe

- Published the first “proof” in 1879
- Proof refuted by Heawood in 1890
- Kempe’s ideas used in the ultimate solution

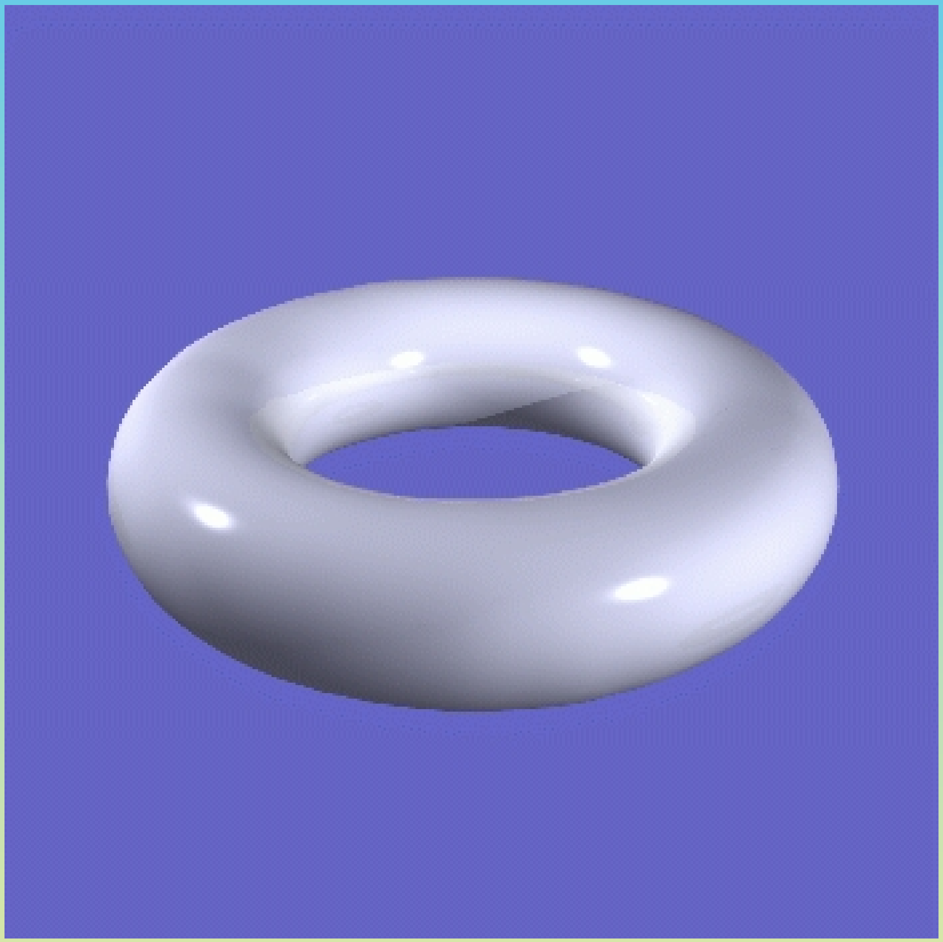
# Percy Heawood

- Refuted Kempe's proof
- Studied colorings of graphs on surfaces

# THE TORUS



# THE TORUS

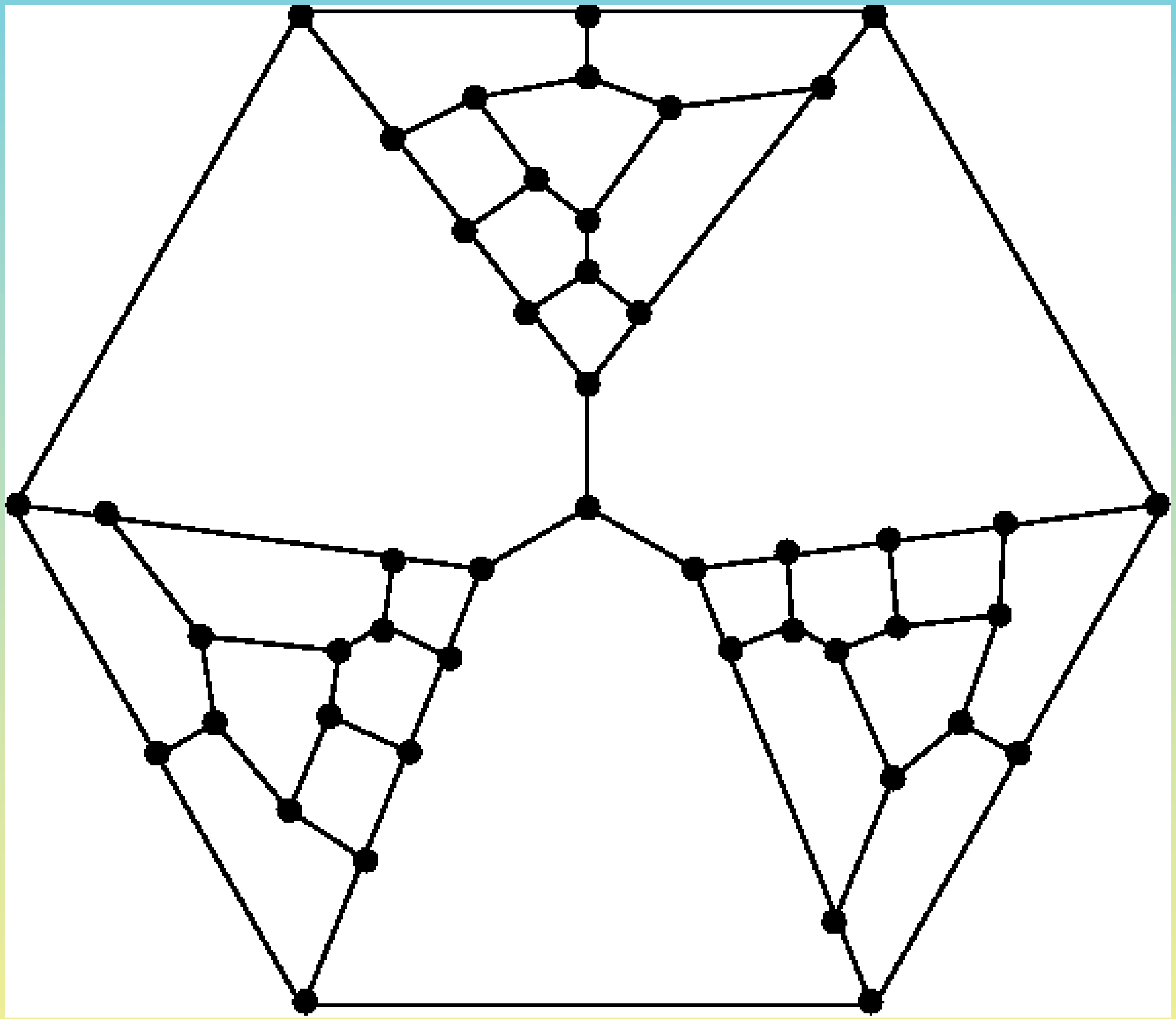


Heawood's formula: $(7+\sqrt{48g+1})/2$ colors suffice

# Peter Guthrie Tait

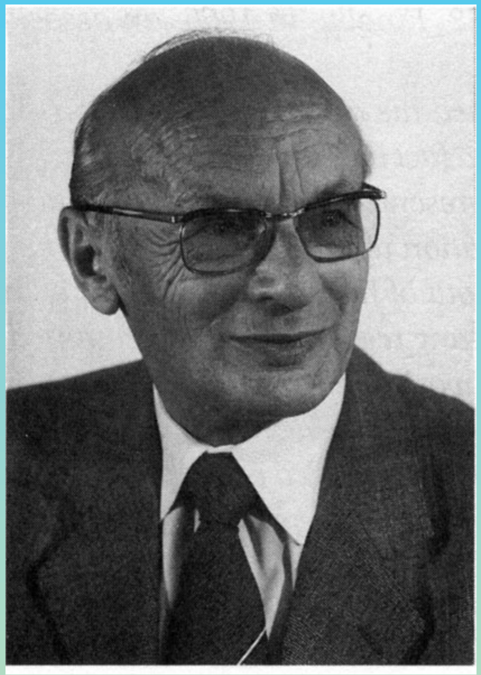
- Published a “proof” in 1880
- Proof refuted by Petersen in 1891 and
- definitely by Tutte in 1954
- Tait’s proof gives an equivalent formulation of the 4CT

# Tutte's example



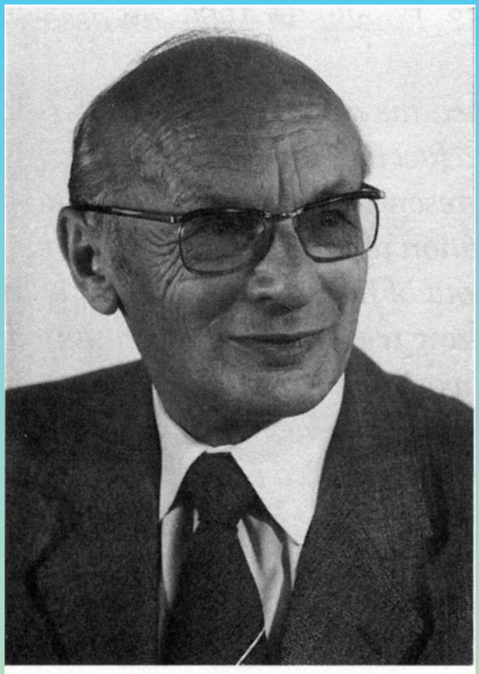
# REDUCIBILITY

developed by Franklin, Bernhard and Bernhard, Reynolds, Winn, Ore and Stemple, Ore, Stromquist, Meyer, Tutte, Whitney, Allaire, Swart, Dūre, Heesch, Miehe



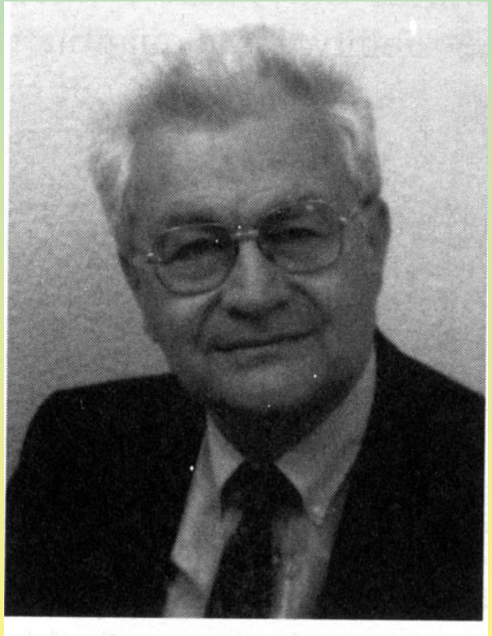
# Heinrich Heesh

Invented discharging



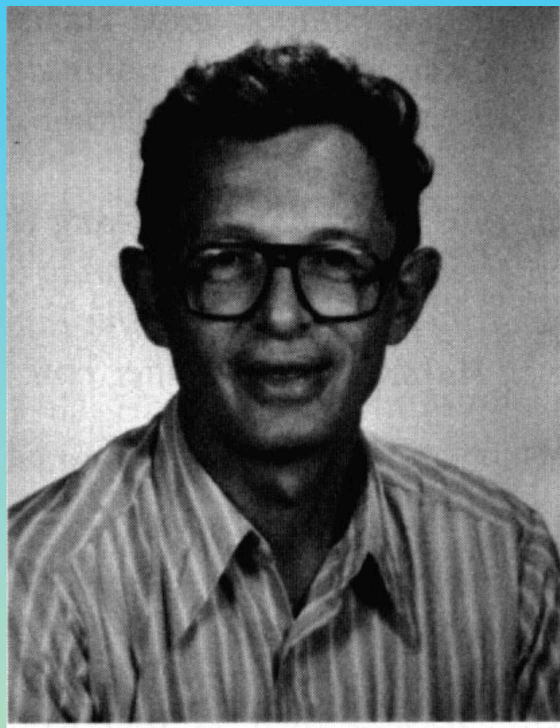
# Heinrich Heesh

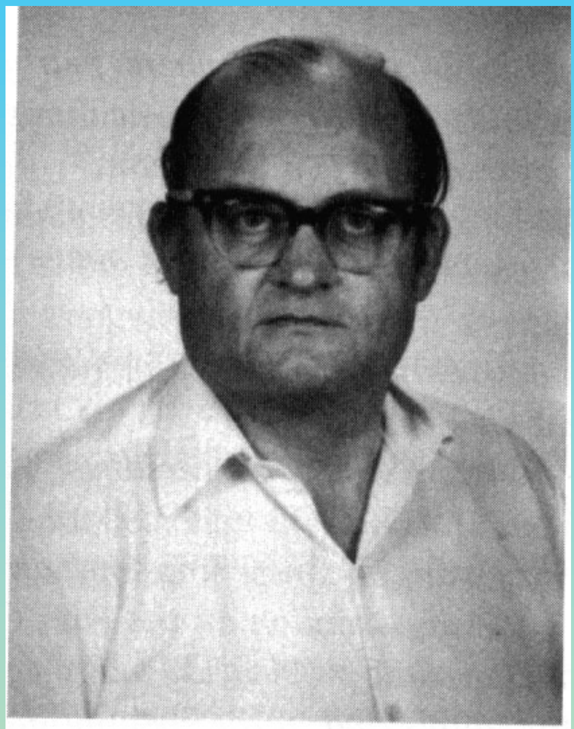
Invented discharging



# Jean Meyer

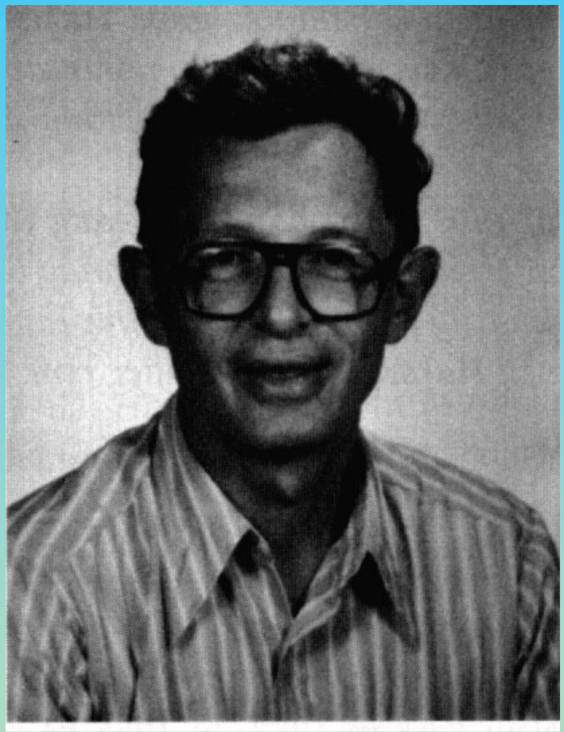
Developed a nice discharging procedure

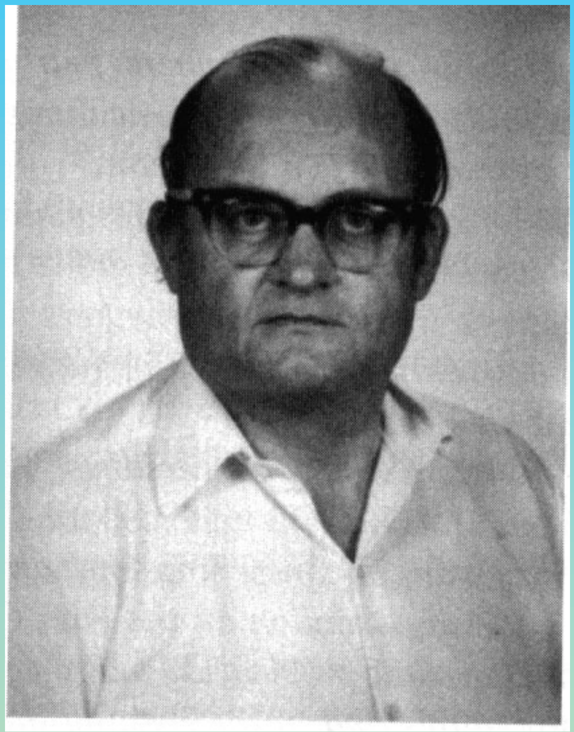




# Kenneth Appel   Wolfgang Haken

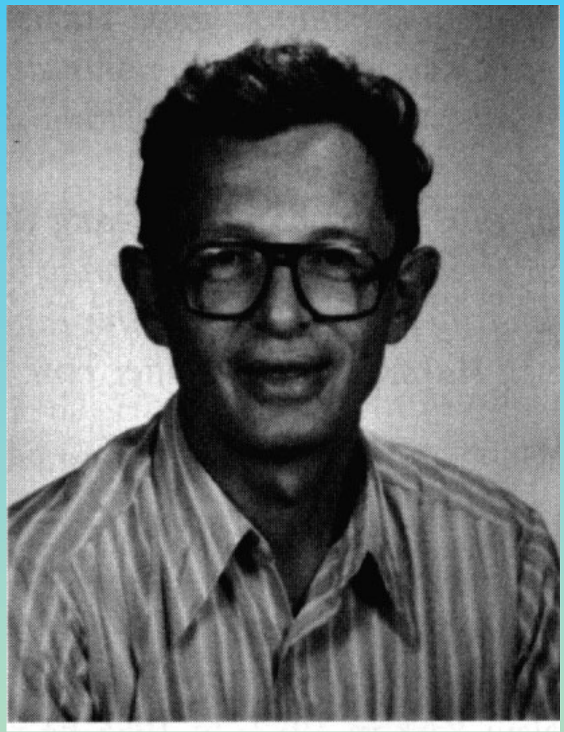
- Published a proof in 1976

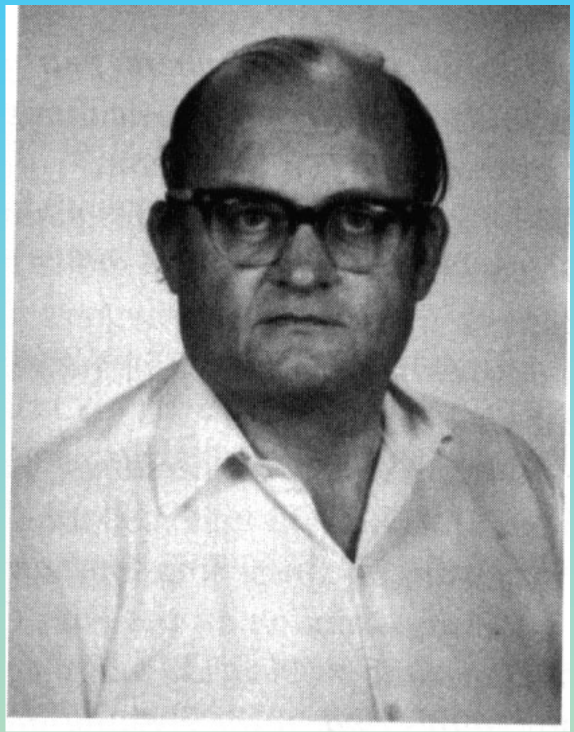




# Kenneth Appel Wolfgang Haken

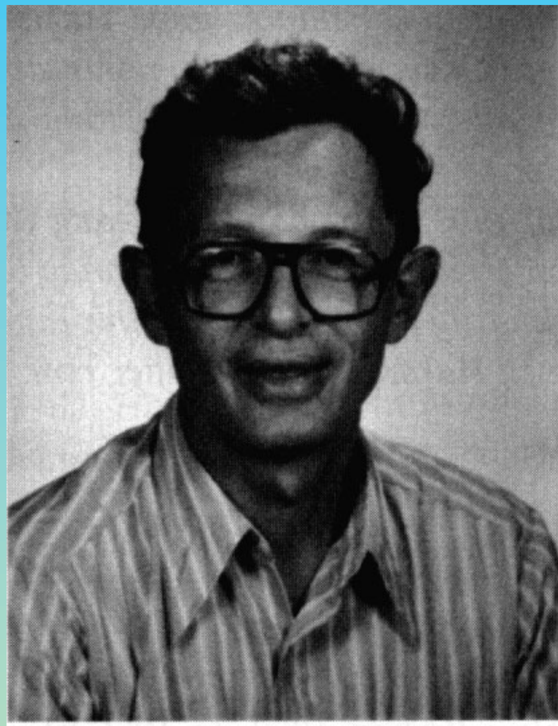
- Published a proof in 1976
- Proof uses computers

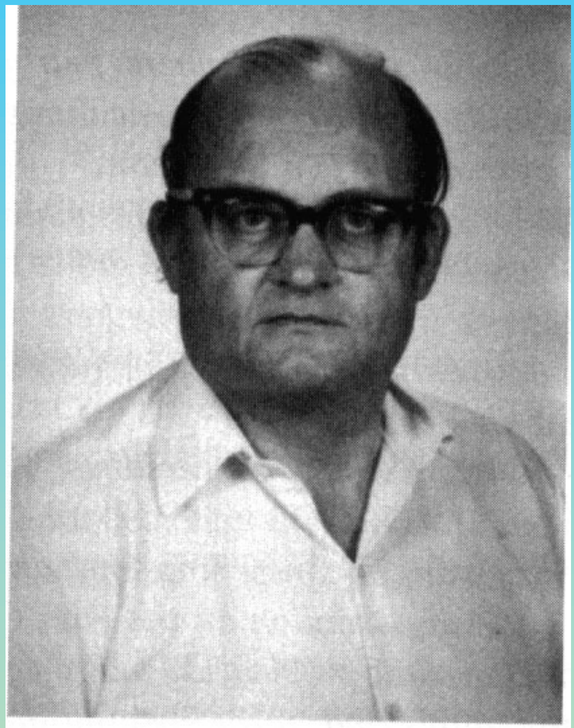




# Kenneth Appel   Wolfgang Haken

- Published a proof in 1976
- Proof uses computers
- Proof is extremely complicated





# Kenneth Appel Wolfgang Haken

- Published a proof in 1976
- Proof uses computers
- Proof is extremely complicated
- Scepticism remains

# EQUIVALENT FORMULATIONS

The cross-product is not associative, and so

$$v_1 \times v_2 \times \cdots \times v_n \tag{1}$$

is not well-defined if $n > 2$. If we put in enough brackets to make it well-defined, we get a bracketing.

The cross-product is not associative, and so

$$v_1 \times v_2 \times \cdots \times v_n \qquad (1)$$

is not well-defined if $n > 2$. If we put in enough brackets to make it well-defined, we get a bracketing.

EXAMPLE $((v_1 \times v_2) \times (v_3 \times v_4)) \times v_5$.

The cross-product is not associative, and so

$$v_1 \times v_2 \times \cdots \times v_n \qquad (1)$$

is not well-defined if $n > 2$. If we put in enough brackets to make it well-defined, we get a bracketing.

EXAMPLE $((v_1 \times v_2) \times (v_3 \times v_4)) \times v_5$.

THEOREM (Kauffman) For every two bracketings of $(1)$ there exists an assignment $v_1, \ldots, v_n \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that the evaluations of the two bracketings are equal and nonzero.

The cross-product is not associative, and so

$$v_1 \times v_2 \times \cdots \times v_n \qquad (1)$$

is not well-defined if $n > 2$. If we put in enough brackets to make it well-defined, we get a bracketing.

EXAMPLE $((v_1 \times v_2) \times (v_3 \times v_4)) \times v_5$.

THEOREM (Kauffman) For every two bracketings of (1) there exists an assignment $v_1, \ldots, v_n \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ such that the evaluations of the two bracketings are equal and nonzero.

THEOREM (Kauffman) The above theorem is equivalent to the 4CT.

THEOREM (Matiyasevich) There exist linear functions $A_k, B_k, C_k, D_k$ $(k = 1, .., 986)$ of $21$ variables such that the 4CT is equivalent to the assertion that for every two integers $n, m$ there exist integers $c_1, \ldots, c_{20}$ such that

$$\prod_{k=1}^{986} \binom{A_k(m, c_1, .., c_{20}) + 7^n B_k(m, c_1, .., c_{20})}{C_k(m, c_1, .., c_{20}) + 7^n D_k(m, c_1, .., c_{20})}$$

is not divisible by $7$.

"While it has sometimes been said that the four color problem is an isolated problem in mathematics, we have found that just the opposite is the case. The four color problem . . . is central to the intersection of algebra, topology, and statistical mechanics."

L. Kauffman and H. Saleur

Comm. Math. Physics

# NEW PROOF

# Appel and Haken, 1986:

"This leaves the reader to face 50 pages containing text and diagrams, 85 pages filled with almost 2500 additional diagrams, and 400 microfiche pages that contain further diagrams and thousands of individual verifications of claims made in the 24 lemmas in the main sections of text. In addition, the reader is told that certain facts have been verified with the use of about twelve hundred hours of computer time and would be extremely time-consuming to verify by hand. The papers are somewhat intimidating due to their style and length and few mathematicians have read them in any detail."

- To my knowledge, no mathematician verified the Appel-Haken proof

- To my knowledge, no mathematician verified the Appel-Haken proof

- Ulrich Schmidt verified $40\%$ of Part I of A&H proof

- To my knowledge, no mathematician verified the Appel-Haken proof

- Ulrich Schmidt verified $40\%$ of Part I of A&H proof

- Schmidt and others found errors that were subsequently corrected by Appel and Haken

# NEW PROOF

Robertson, Sanders, Seymour, Thomas, published in 1997

# NEW PROOF

Robertson, Sanders, Seymour, Thomas, published in 1997

- Still computer assisted

# NEW PROOF

Robertson, Sanders, Seymour, Thomas, published in 1997

- Still computer assisted
- Article has 42 pages

# NEW PROOF

Robertson, Sanders, Seymour, Thomas, published in 1997

- Still computer assisted
- Article has 42 pages
- Programs available for independent verification

# NEW PROOF

Robertson, Sanders, Seymour, Thomas, published in 1997

- Still computer assisted
- Article has 42 pages
- Programs available for independent verification
- Independent programs written by Gašper Fijavž

# NEW PROOF

Robertson, Sanders, Seymour, Thomas, published in 1997

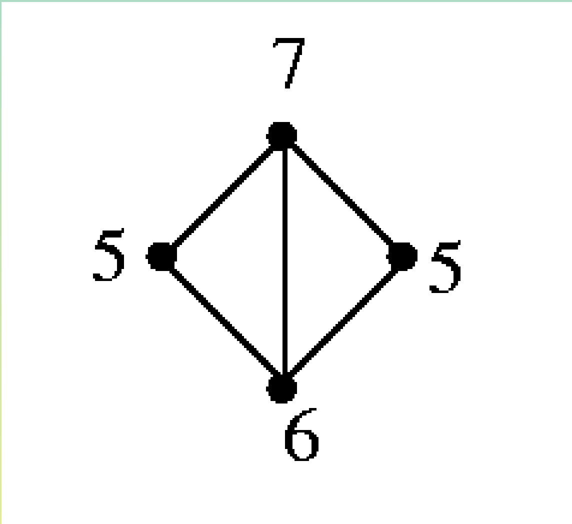
- Still computer assisted
- Article has 42 pages
- Programs available for independent verification
- Independent programs written by Gašper Fijavž

COROLLARY A quadratic algorithm to 4-color planar graphs.

# OUTLINE OF PROOF

Let $G$ be a counterexample with $|V(G)|$ minimum. It follows that $G$ is an internally 6-connected triangulation.

A configuration:



We exhibit a set $\mathcal{U}$ of $633$ configurations such that

We exhibit a set $\mathcal{U}$ of $633$ configurations such that

THEOREM 1 No member of $\mathcal{U}$ "appears" in a minimal counterexample to the 4CT

We exhibit a set $\mathcal{U}$ of $633$ configurations such that

THEOREM 1 No member of $\mathcal{U}$ "appears" in a minimal counterexample to the 4CT

THEOREM 2 For every internally 6-connected triangulation $T$, some member of $\mathcal{U}$ "appears" in $T$.

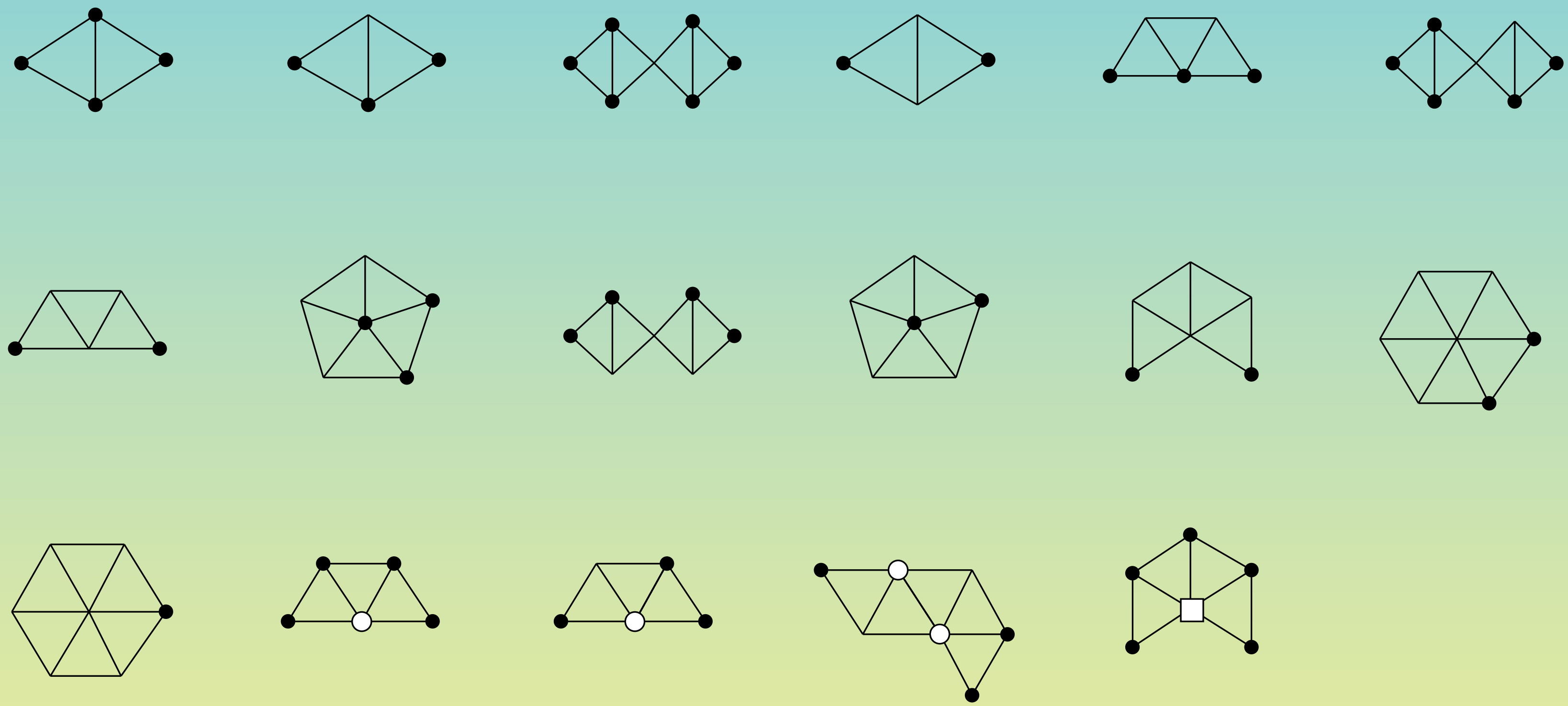
We exhibit a set $\mathcal{U}$ of $633$ configurations such that

THEOREM 1 No member of $\mathcal{U}$ "appears" in a minimal counterexample to the 4CT

THEOREM 2 For every internally 6-connected triangulation $T$, some member of $\mathcal{U}$ "appears" in $T$.

A configuration $K$ appears in a triangulation $T$ if $K$ is an induced subgraph of $T$ and for every vertex of $K$ its label equals its degree in $T$.

# A SUBSET OF $\mathcal{U}$



# ABOUT THEOREM 1

THEOREM 1 No member of $\mathcal{U}$ appears in a minimal counterexample to the 4CT

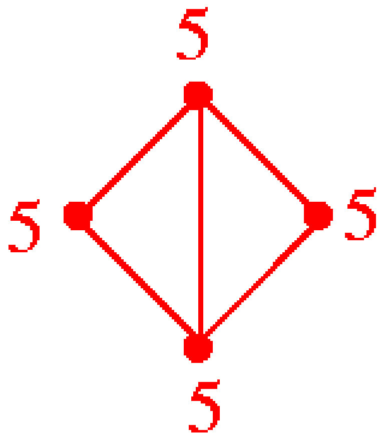
PROOF Suppose one of them does.

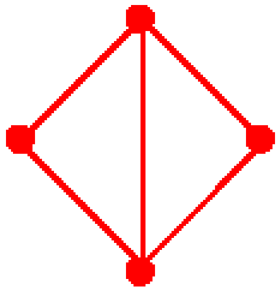
# ABOUT THEOREM 1

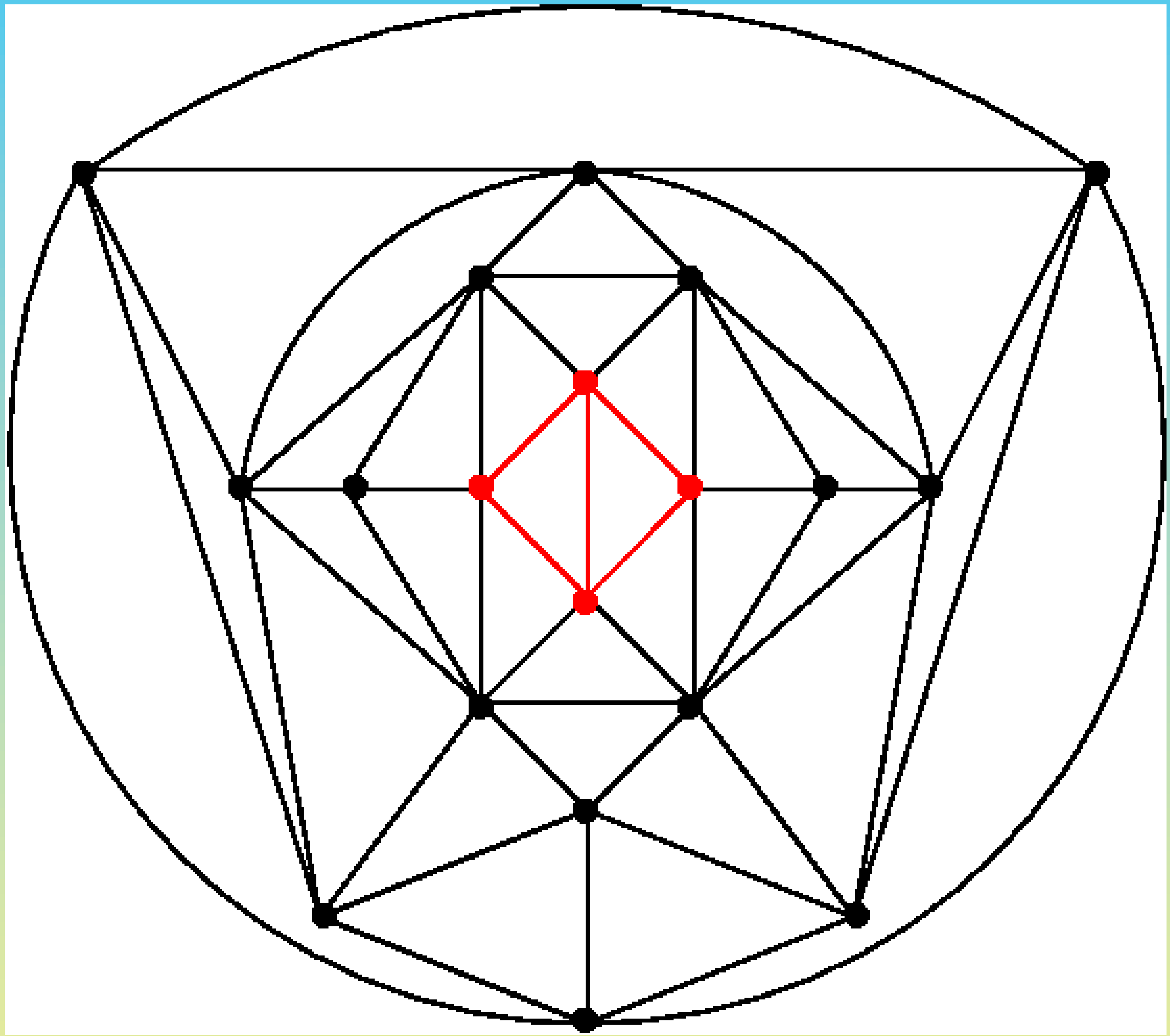
THEOREM 1 No member of $\mathcal{U}$ appears in a minimal counterexample to the 4CT

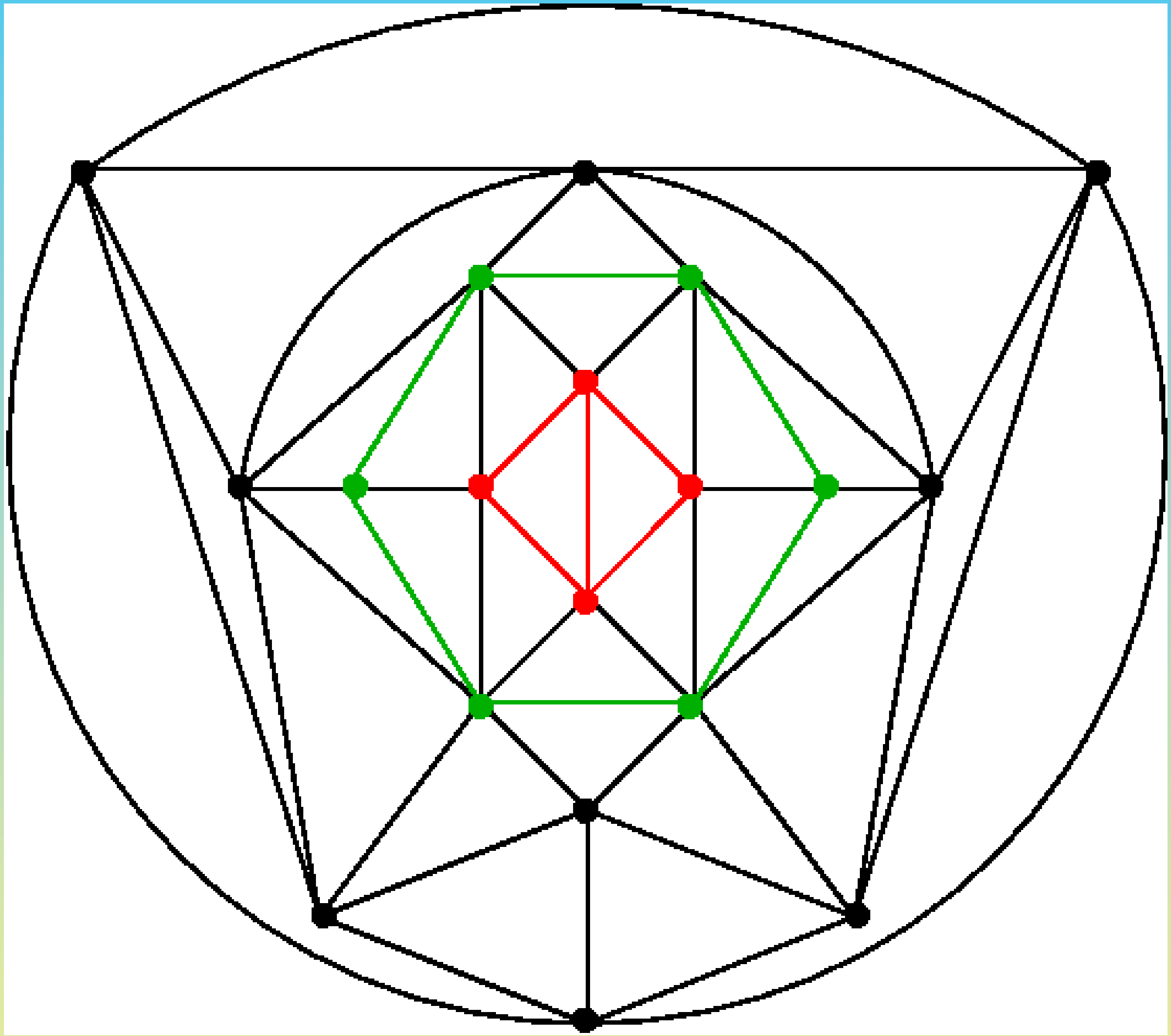
PROOF Suppose one of them does.

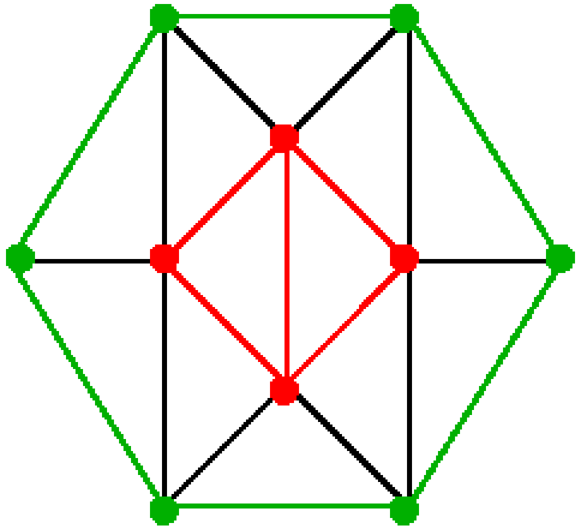
REMINDER A configuration $K$ appears in a triangulation $T$ if $K$ is an induced subgraph of $T$ and for every vertex of $K$ its label equals its degree in $T$.

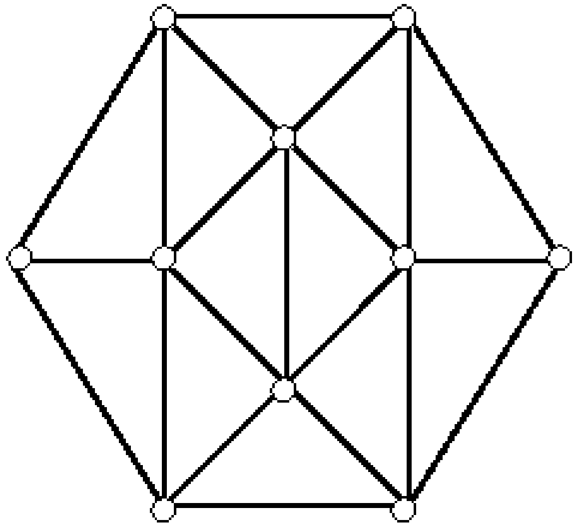

5
5
5
5

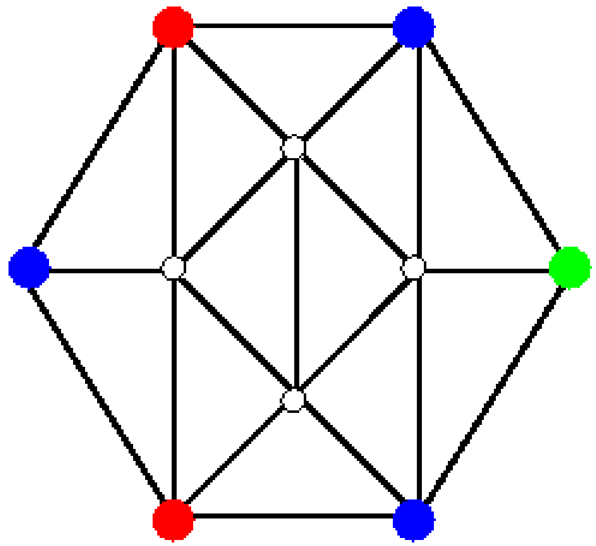


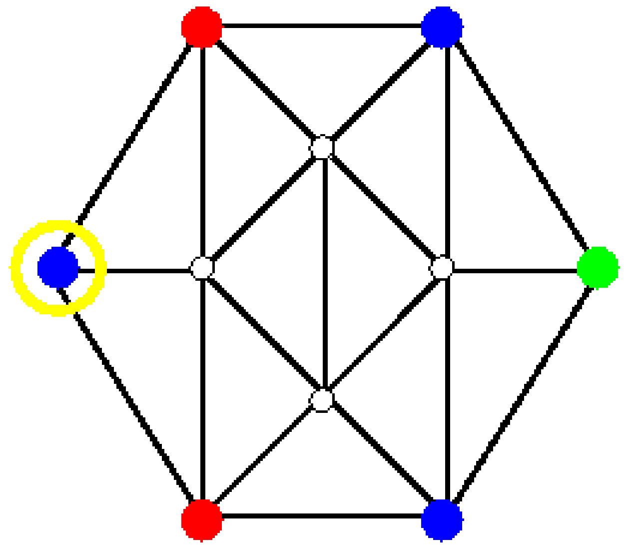


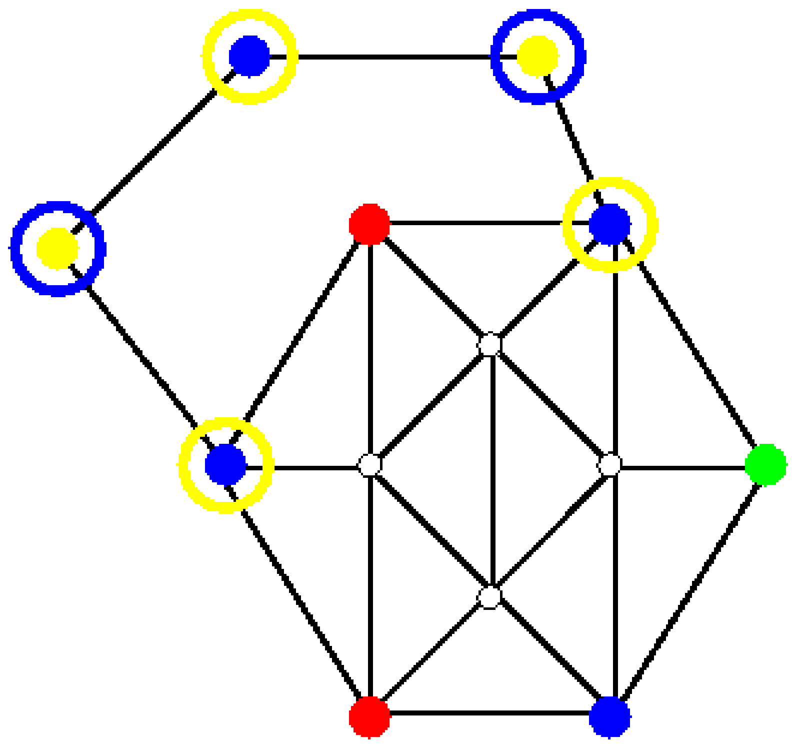


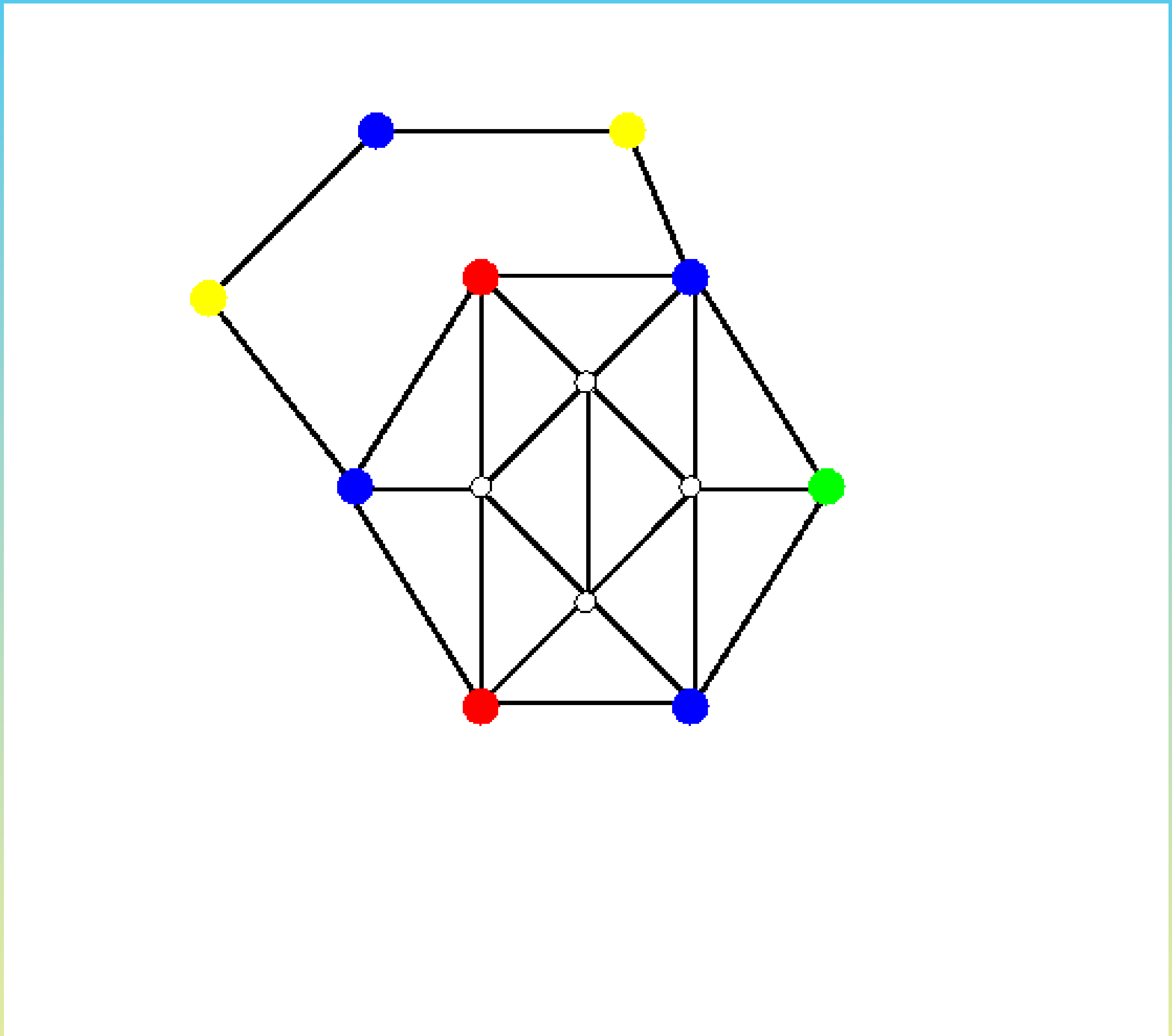


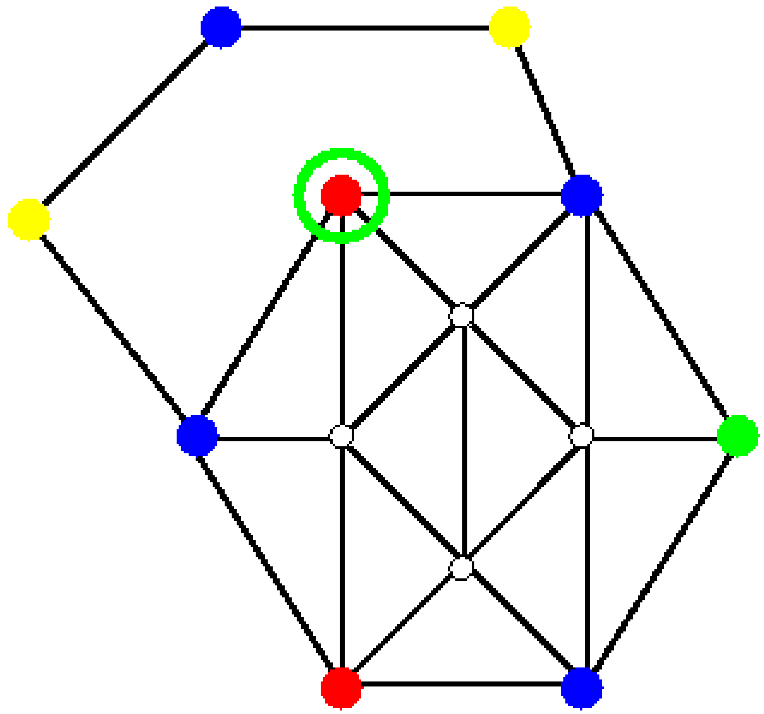


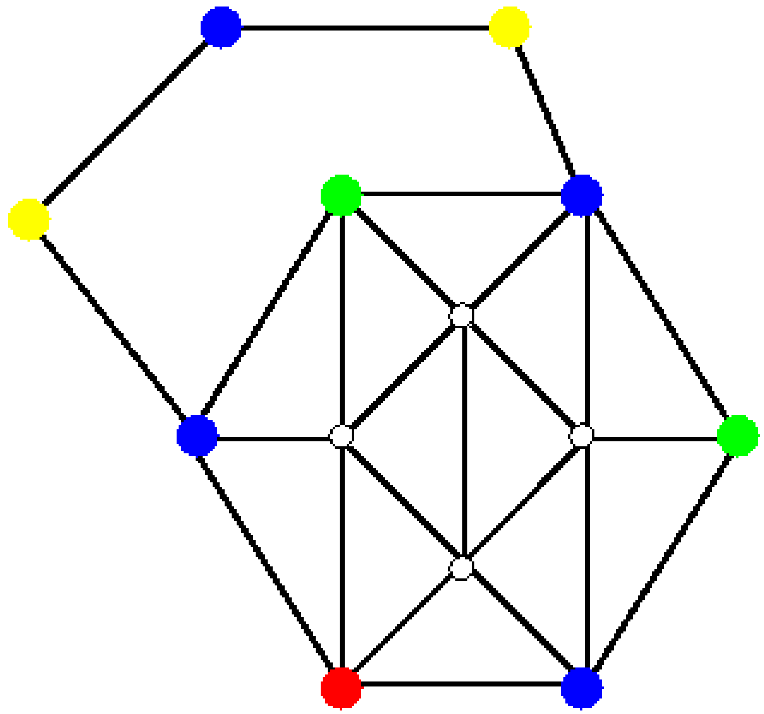


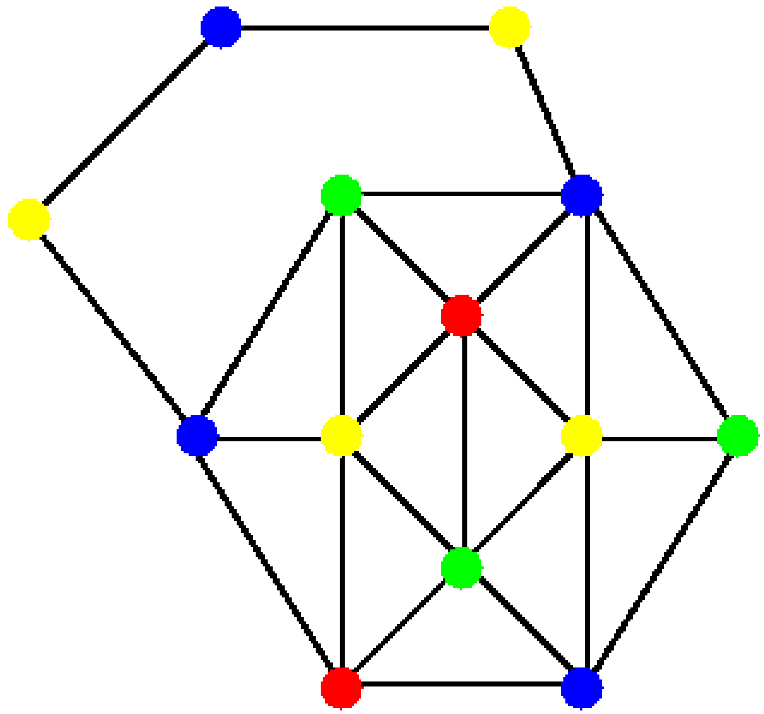


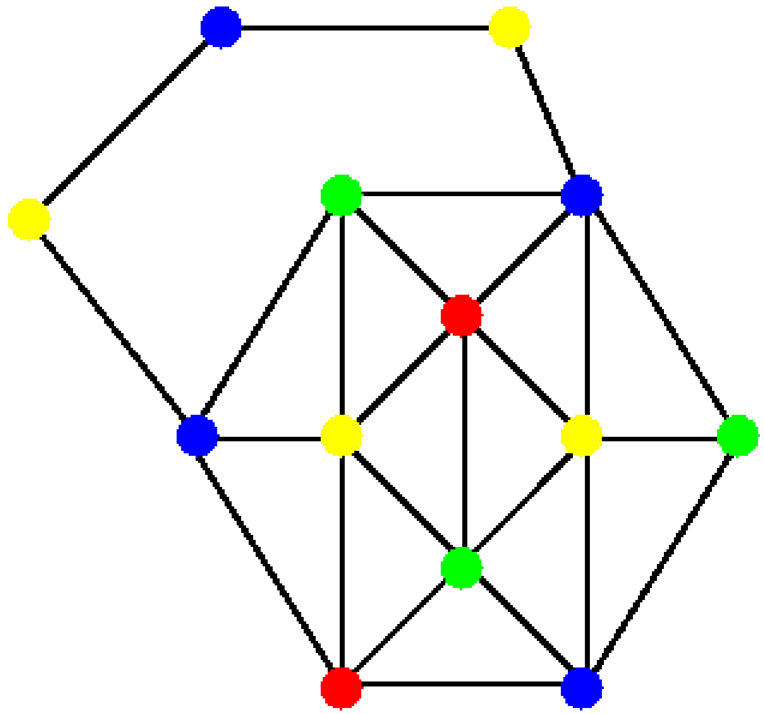




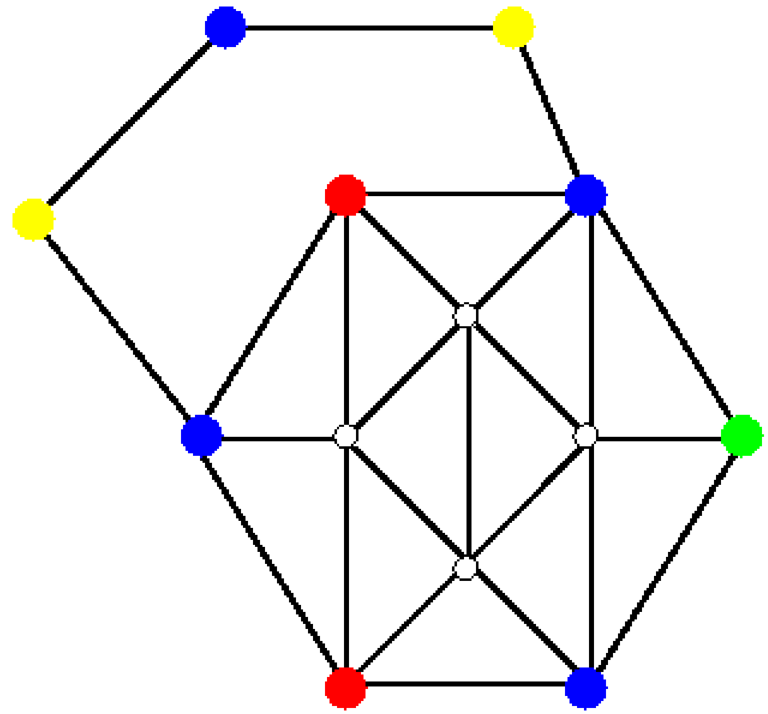


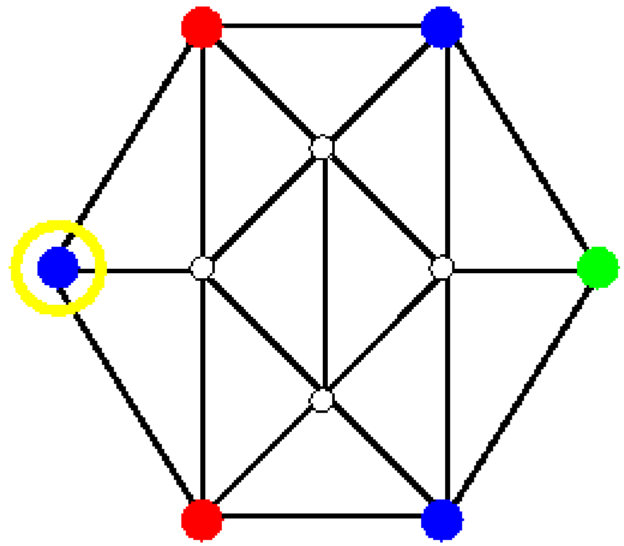


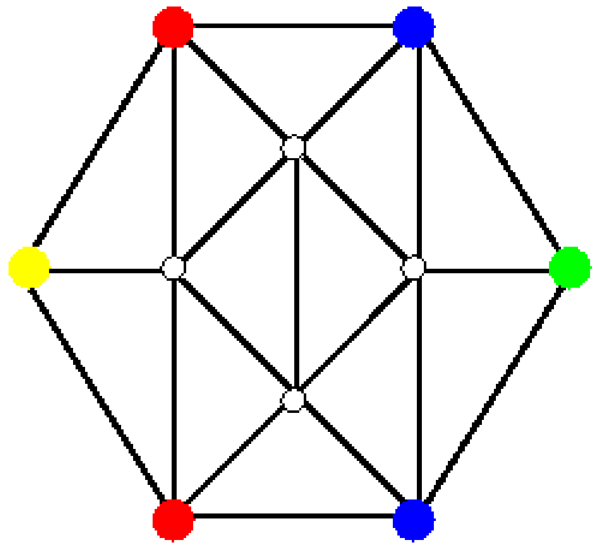


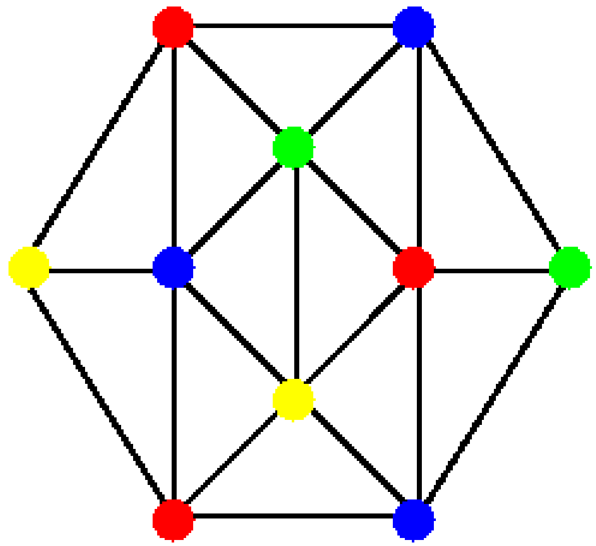


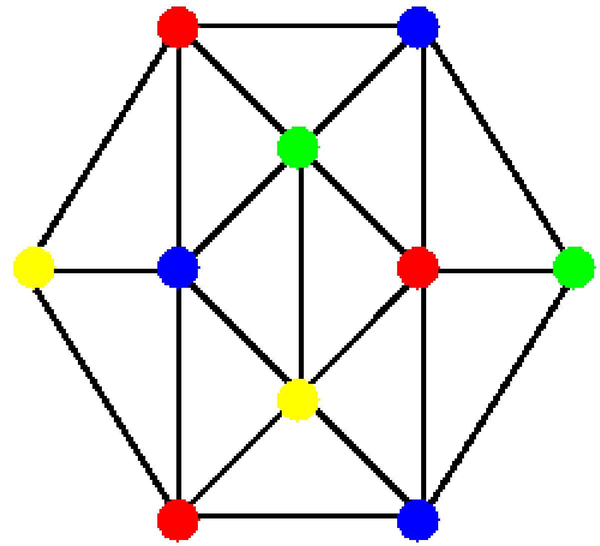
This gives a coloring of the entire graph, a contradiction.











This gives a coloring of the entire graph, a contradiction.

This procedure can be automated and carried out on a computer. In fact, it must be carried out on a computer, because for configurations with rings of size $14$ there are almost $200,000$ colorings to consider.

THEOREM 2 For every internally 6-connected triangulation $T$, some member of $\mathcal{U}$ appears in $T$.

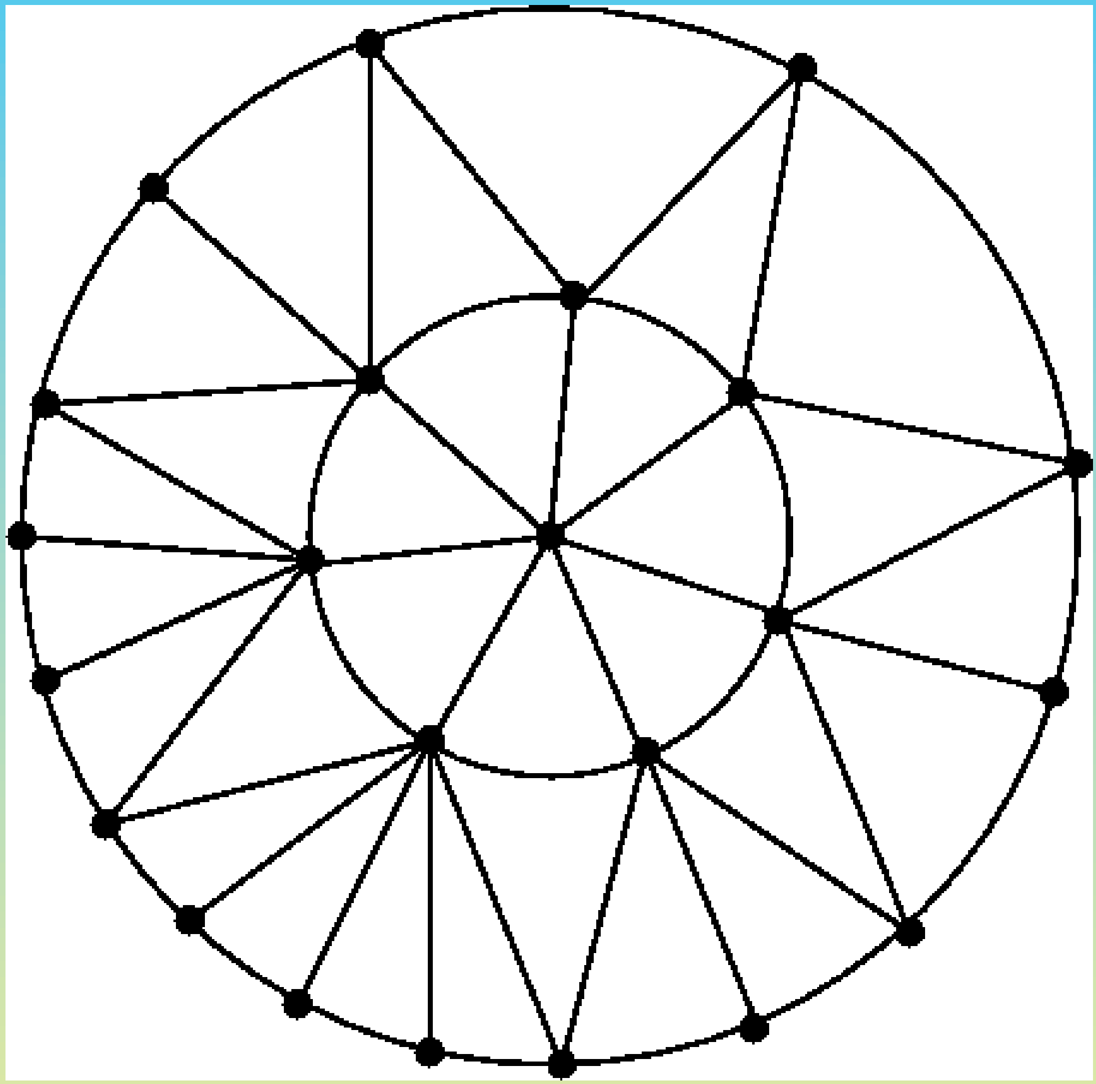
PROOF In a triangulation $e = 3n - 6$ and so

$$d_1 + d_2 + \cdots + d_n = 2e = 6n - 12$$

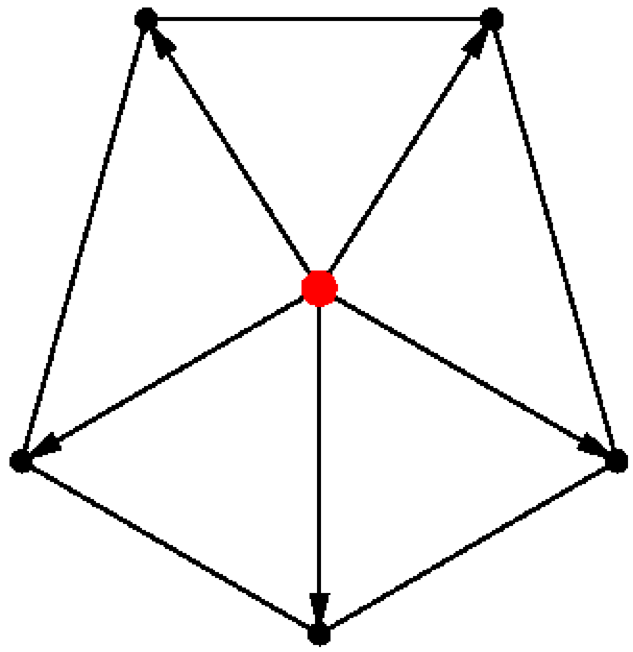
or

$$(6 - d_1) + (6 - d_2) + \cdots + (6 - d_n) = 12.$$

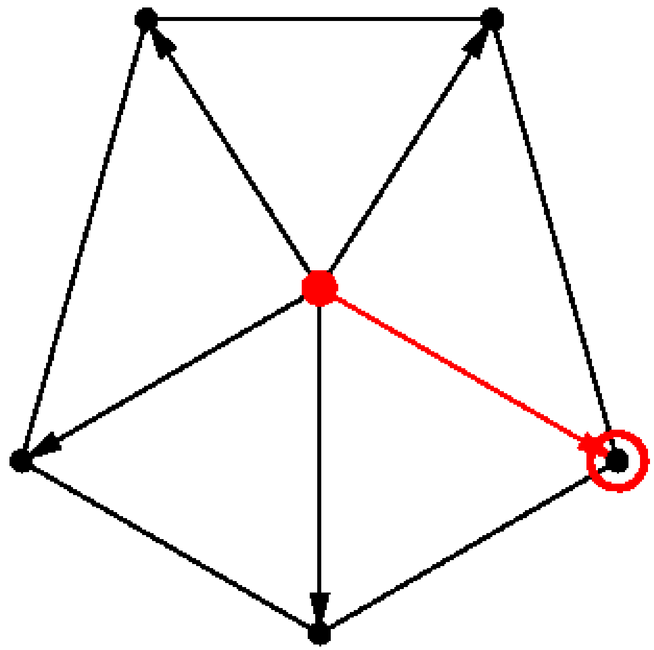
Initially, a vertex of degree $d$ will receive a charge of $10(6 - d)$. Thus the sum of the charges is $120$. Charges will be redistributed according to certain rules, but the total sum will remain the same. Thus there is a vertex $v$ of positive charge. We show that a member of $\mathcal{U}$ appears in the second neighborhood of $v$.



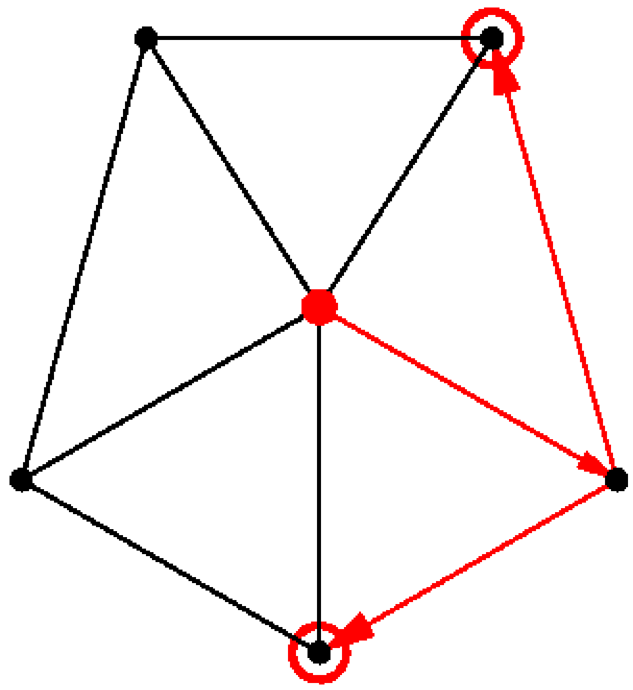
initial charge of $v$ is $10(6-\text{deg}(v))$



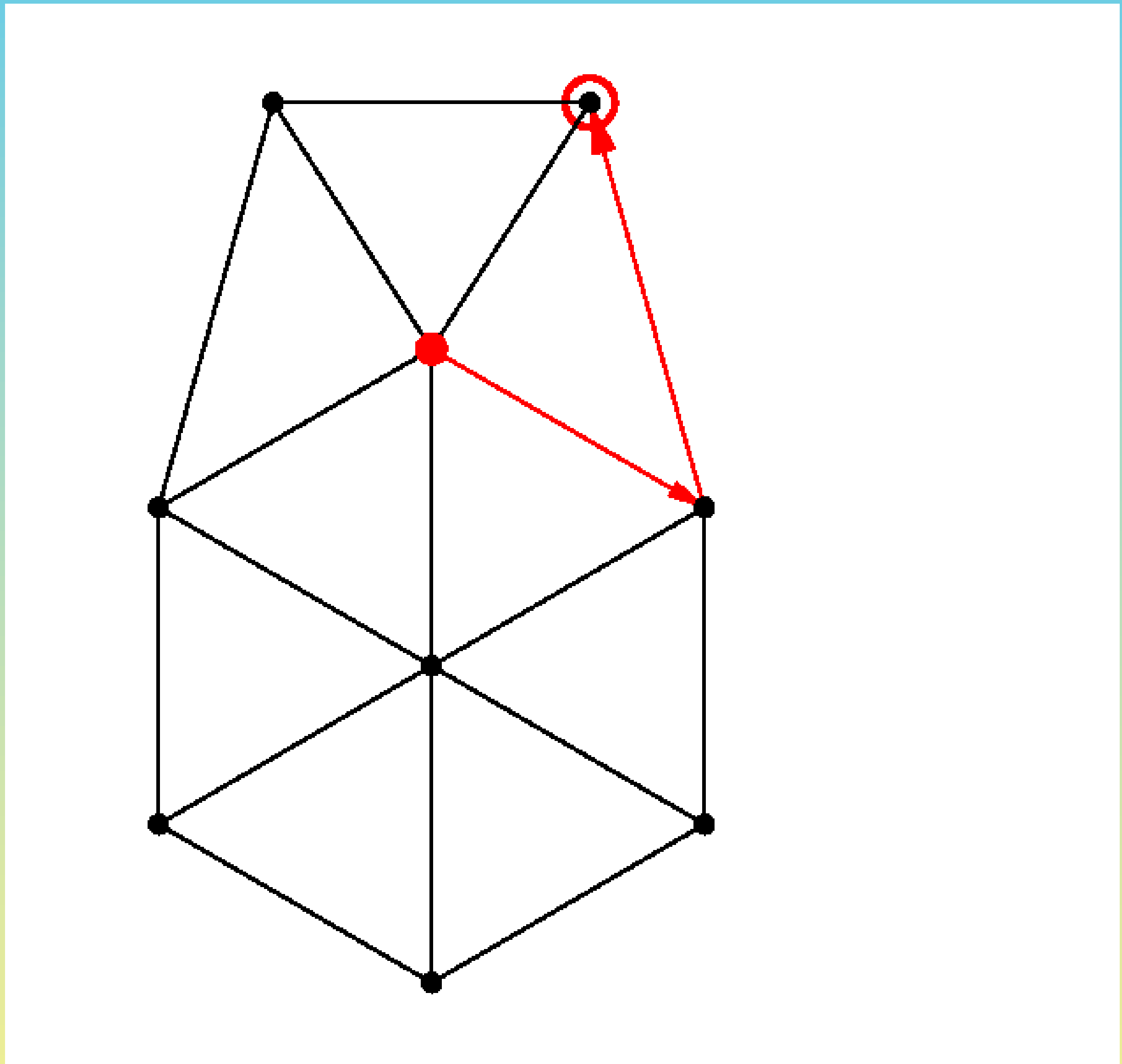
# initial charge of $v$ is $10(6-\deg(v))$



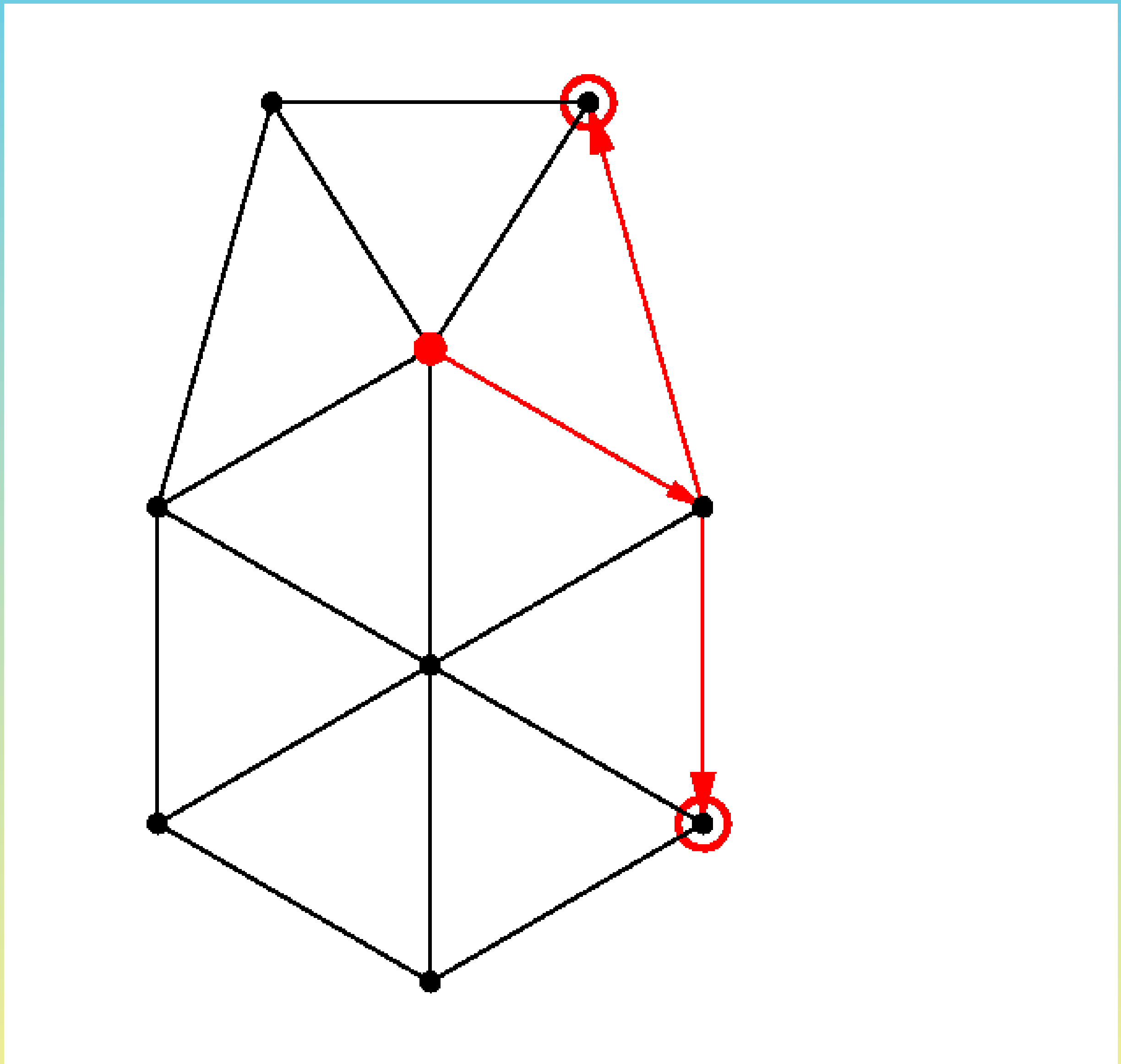
initial charge of $v$ is $10(6-\deg(v))$



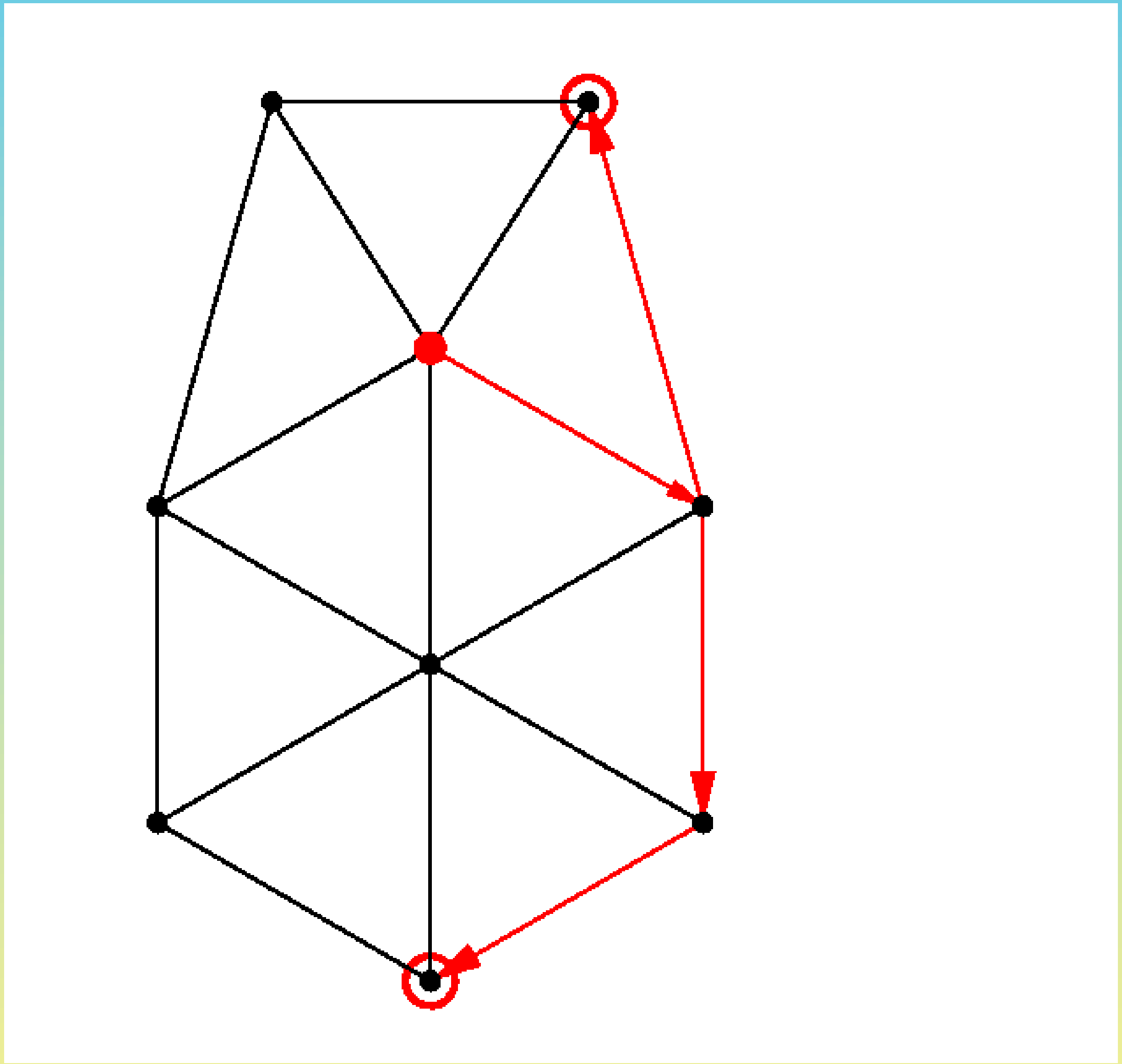
# initial charge of $v$ is $10(6-\mathsf{deg}(v))$



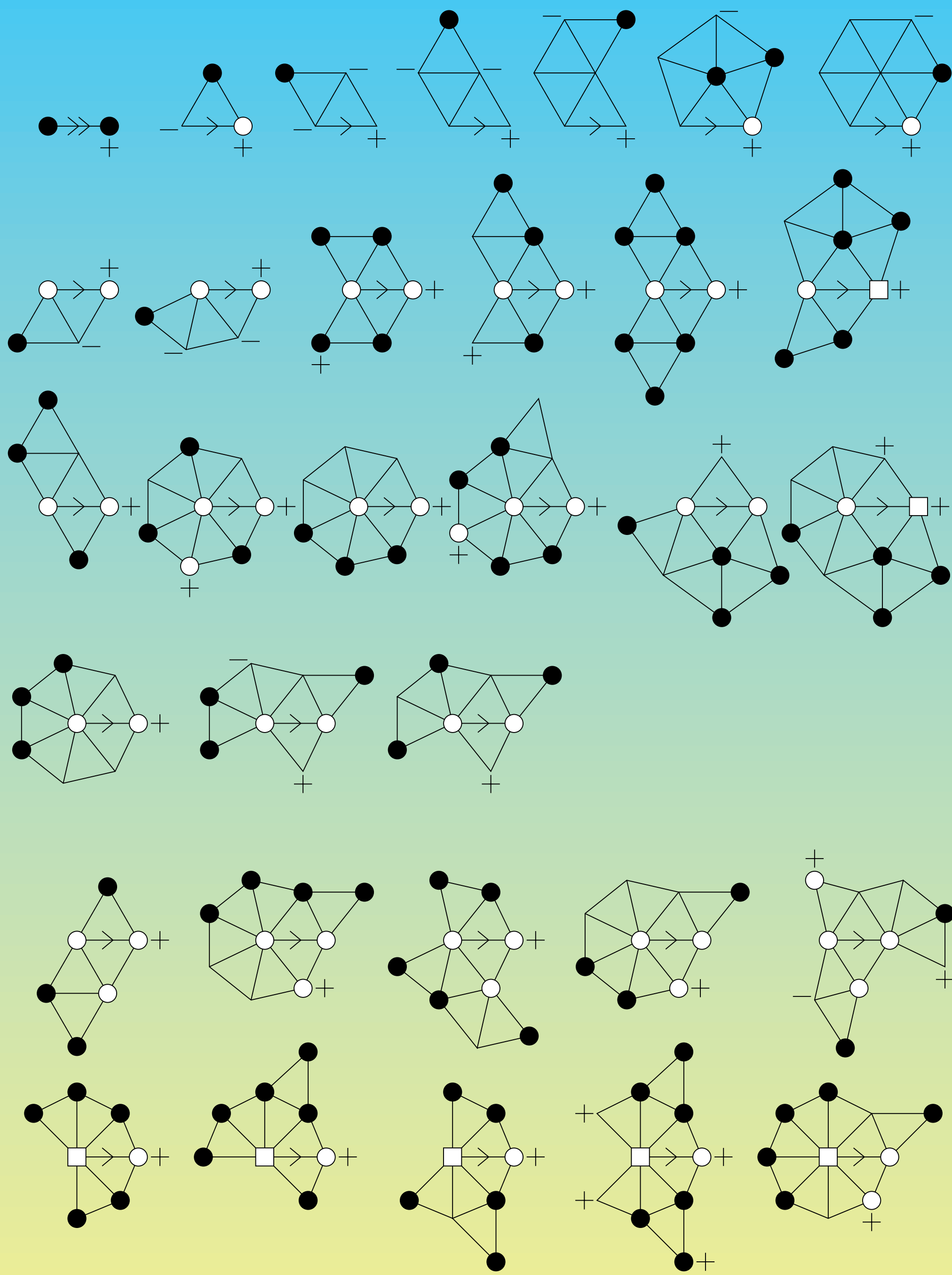
# initial charge of $v$ is $10(6-\deg(v))$



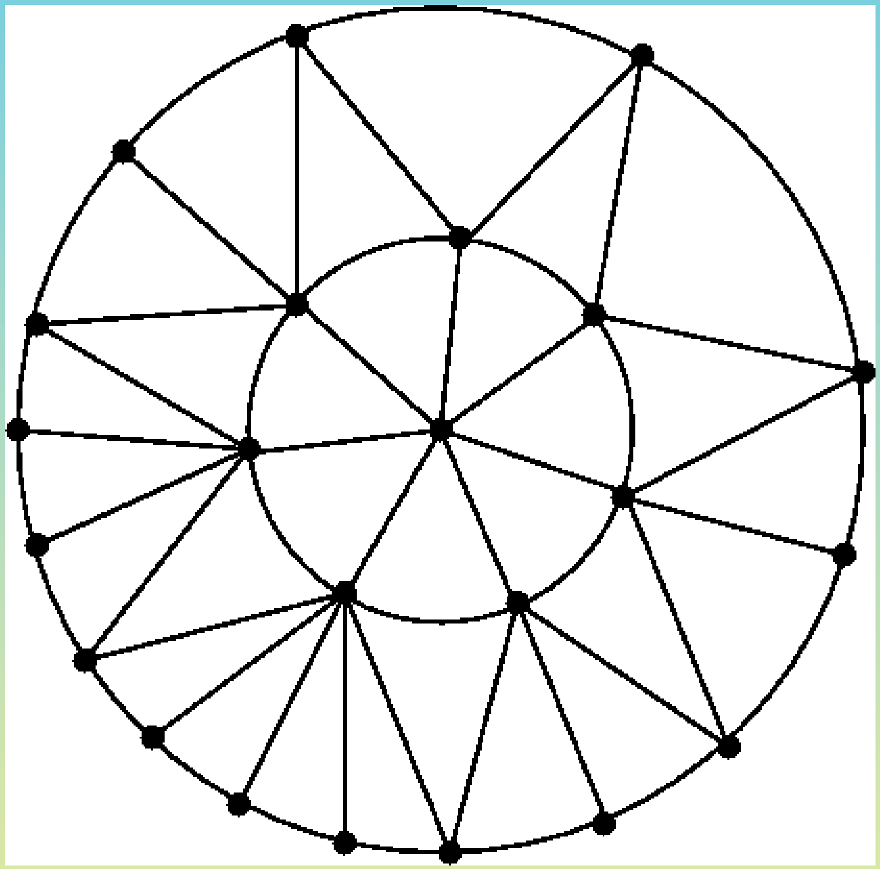
# initial charge of $v$ is $10(6-\deg(v))$



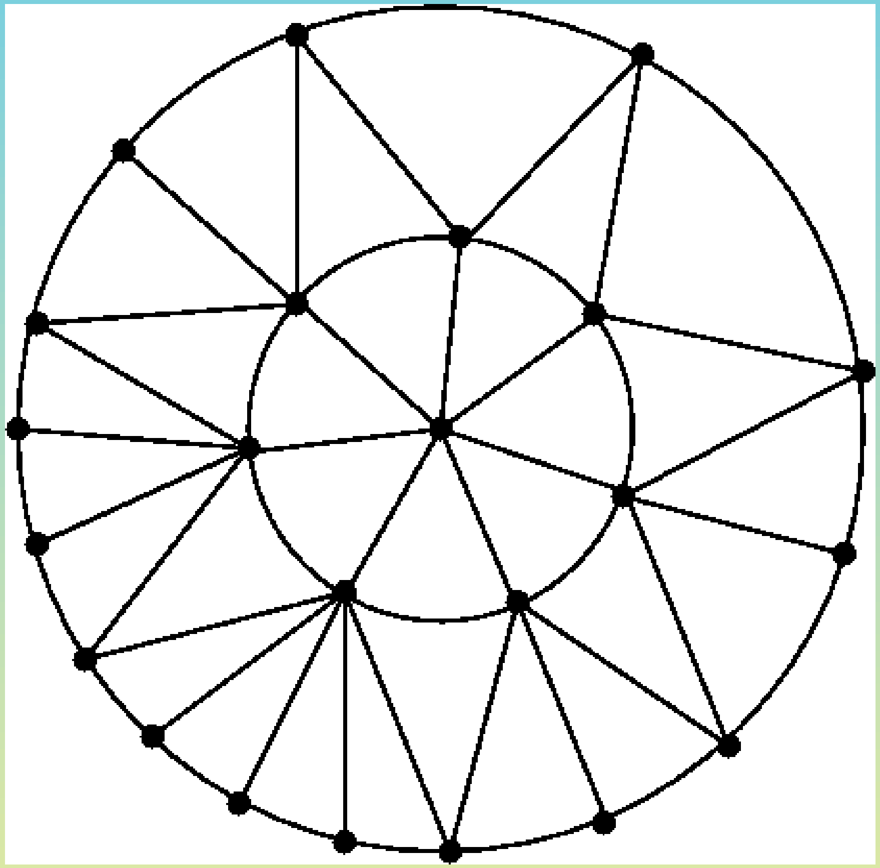
These are the basic discharging rules. In addition, we need $25$ “secondary” rules. The secondary rules have no geometric interpretation, and were designed by trial and error.



The final charge only depends on the second neighborhood of a vertex:



The final charge only depends on the second neighborhood of a vertex:



The program examines all possible second neighborhoods.

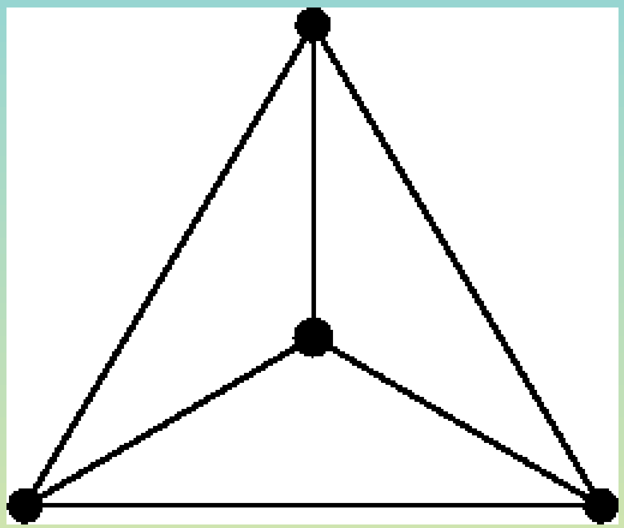
# IS THIS REALLY A PROOF?

# IS THIS REALLY A PROOF?

PROGRAM $\longrightarrow$ COMPILER $\longrightarrow$ HARDWARE $\longrightarrow$ RESULTS

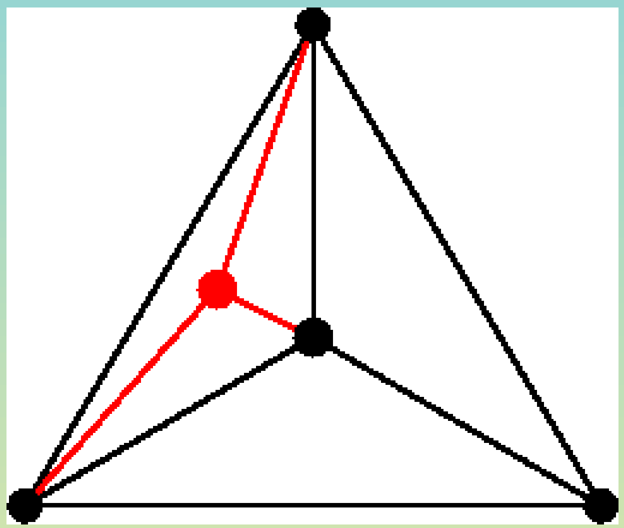
# BEYOND THE 4CT

Uniquely $4$-colorable planar graphs:



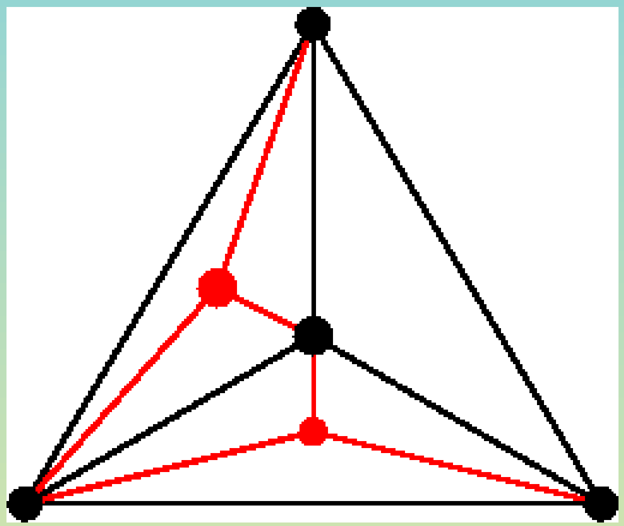
# BEYOND THE 4CT

Uniquely $4$-colorable planar graphs:



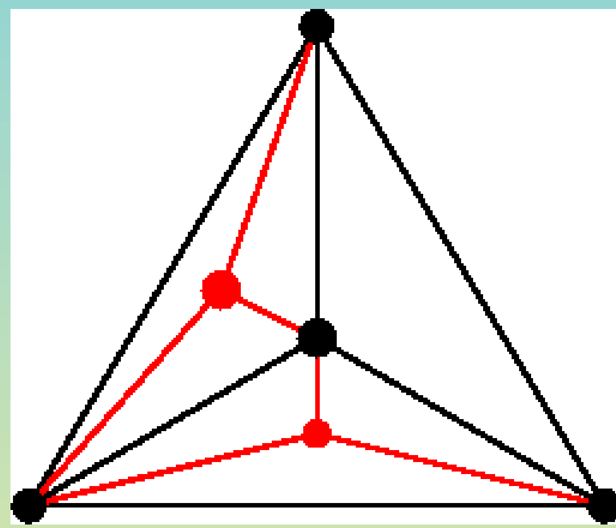
# BEYOND THE 4CT

Uniquely $4$-colorable planar graphs:



# BEYOND THE 4CT

Uniquely $4$-colorable planar graphs:



THEOREM (Fowler) Every planar graph has at least two $4$-colorings, unless it belongs to the above family.

# SUMMARY

- The 4CT inspired a lot of theory and problems

# SUMMARY

- The 4CT inspired a lot of theory and problems
- Over two dozen equivalent formulations (in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras,...)

# SUMMARY

- The 4CT inspired a lot of theory and problems
- Over two dozen equivalent formulations (in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras,...)
- Proof is computer-assisted

# SUMMARY

- The 4CT inspired a lot of theory and problems
- Over two dozen equivalent formulations (in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras,...)
- Proof is computer-assisted
- Conjectured generalizations

# SUMMARY

- The 4CT inspired a lot of theory and problems
- Over two dozen equivalent formulations (in terms of vector cross products, Lie algebras, divisibility, Temperley-Lieb algebras,...)
- Proof is computer-assisted
- Conjectured generalizations
- See August 1998 Notices of the AMS for a survey