PFAFFIAN ORIENTATIONS OF GRAPHS

Robin Thomas

School of Mathematics
Georgia Institute of Technology
http://www.math.gatech.edu/~thomas

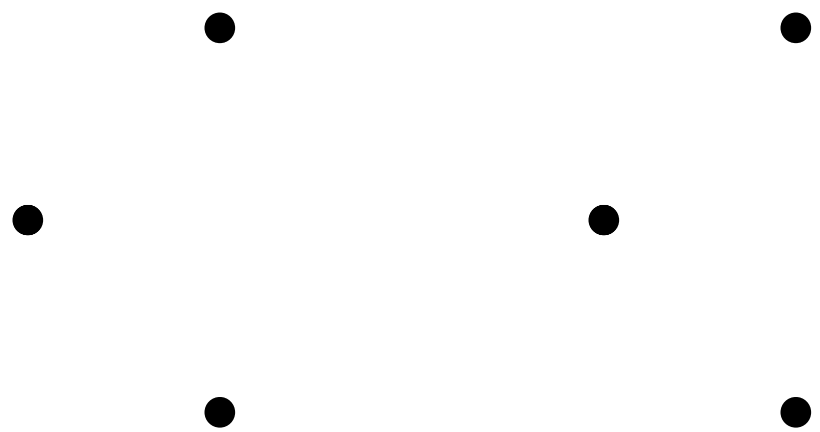
joint work with

Serguei Norine

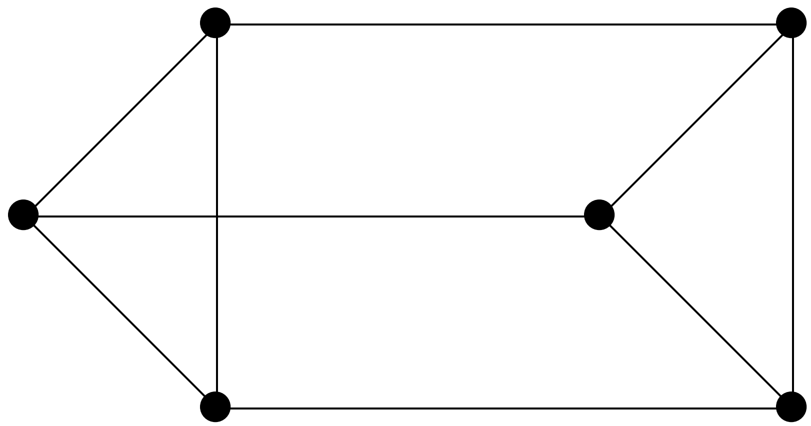
Neil Robertson

P. D. Seymour

Graphs have vertices

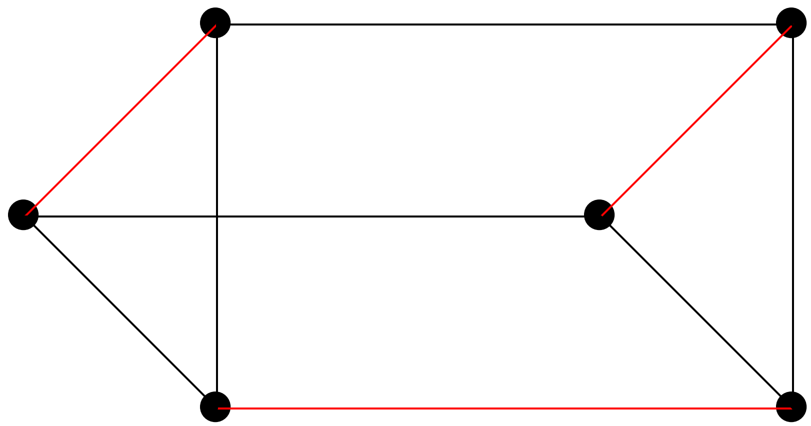


Graphs have vertices and edges



A perfect matching consists of independent edges saturating all the vertices

Graphs have vertices and edges



A perfect matching consists of independent edges saturating all the vertices

The Pfaffian of a skew symmetric matrix A

$$\mathrm{Pf}(A) = \sum sign\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix} a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n}$$

the summation over all partitions $\{\{i_1,j_1\},\{i_2,j_2\},\dots,\{i_n,j_n\}\}$ of $[2n]$ into unordered pairs.

Lemma $\mathrm{Pf}^2(A)=\det(A)$

In particular, $\mathrm{Pf}(A)$ can be computed efficiently.

The Pfaffian of a skew symmetric matrix *A*

$$\mathrm{Pf}(A) = \sum sign\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix} a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n}$$

the summation over all partitions $\underbrace{\{\{i_1,j_1\},\{i_2,j_2\},\dots,\{i_n,j_n\}\}}_{M}$ of $[2n]$ into unordered pairs.

Now let *A* be a skew adjacency matrix of a graph *G*.

Order the pairs (i_k,j_k) to make sure $a_{i_k j_k}=1$. Then

$$\mathrm{Pf}(A) = \sum sign\begin{pmatrix} 1 & 2 & \dots & 2n-1 & 2n \\ i_1 & j_1 & \dots & i_n & j_n \end{pmatrix}$$

$\mathrm{sgn}_D(M)$

The Pfaffian of a skew symmetric matrix A

$$\mathrm{Pf}(A)=\sum sign\begin{pmatrix}1 & 2 & \dots & 2n-1 & 2n\\ i_1 & j_1 & \dots & i_n & j_n\end{pmatrix} a_{i_1j_1} a_{i_2j_2} \dots a_{i_nj_n}$$

the summation over all partitions $\{\{i_1,j_1\},\{i_2,j_2\},\dots,\{i_n,j_n\}\}$ of $[2n]$ into unordered pairs.

Now let A be a skew adjacency matrix of a graph G.

Order the pairs (i_k,j_k) to make sure $a_{i_kj_k}=1$. Then

$$\mathrm{Pf}(A)=\sum sign\begin{pmatrix}1 & 2 & \dots & 2n-1 & 2n\\ i_1 & j_1 & \dots & i_n & j_n\end{pmatrix}$$

$$\mathrm{Pf}(A)=\sum sign\begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix}$$

DEF An orientation *D* of a graph *G* is Pfaffian if $sgn_D(M)=sgn_D(M')$ for every two perfect matchings *M*,*M*'.

In that case the number of perfect matchings can be efficiently calculated.

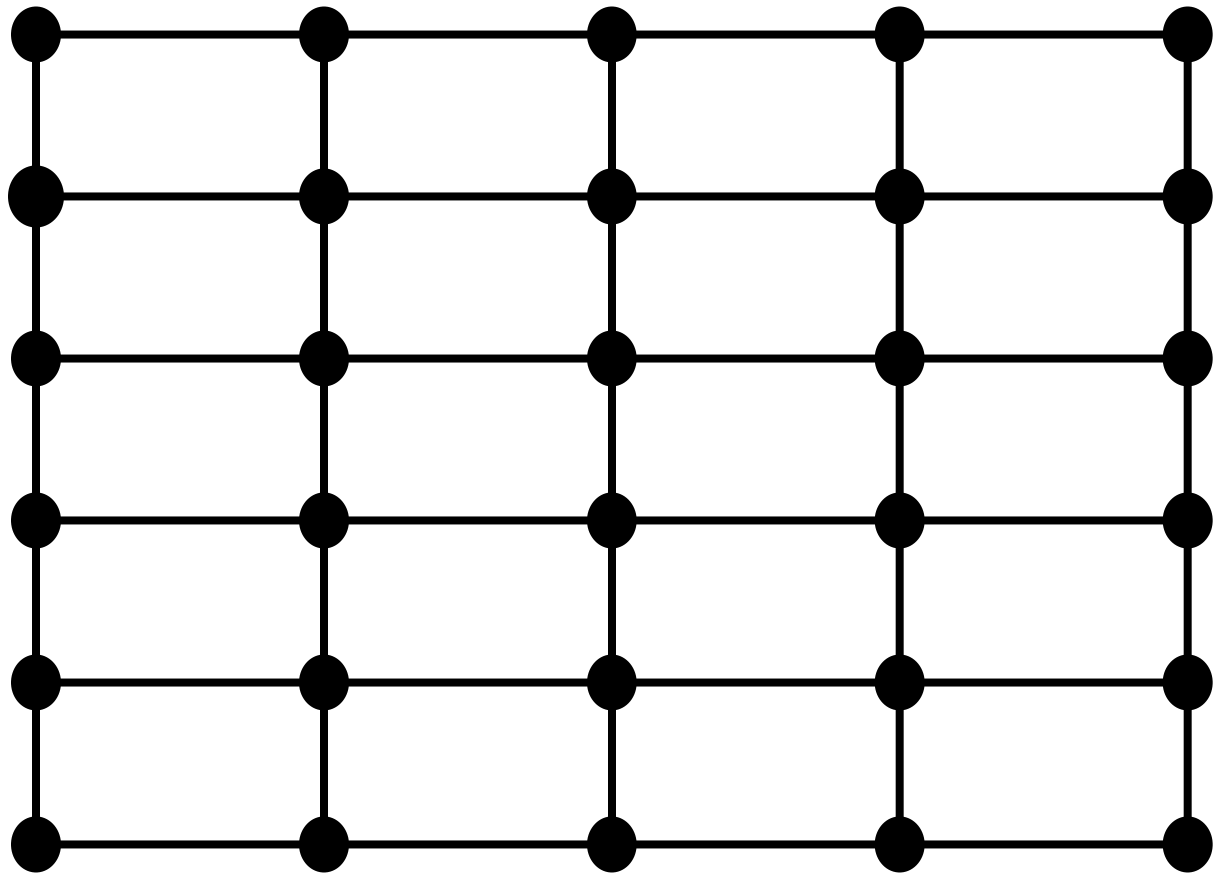
$$\text{Pf}(A) = \sum sign\begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix}$$

DEF An orientation D of a graph G is Pfaffian if $\text{sgn}_D(M)=\text{sgn}_D(M')$ for every two perfect matchings M,M'.

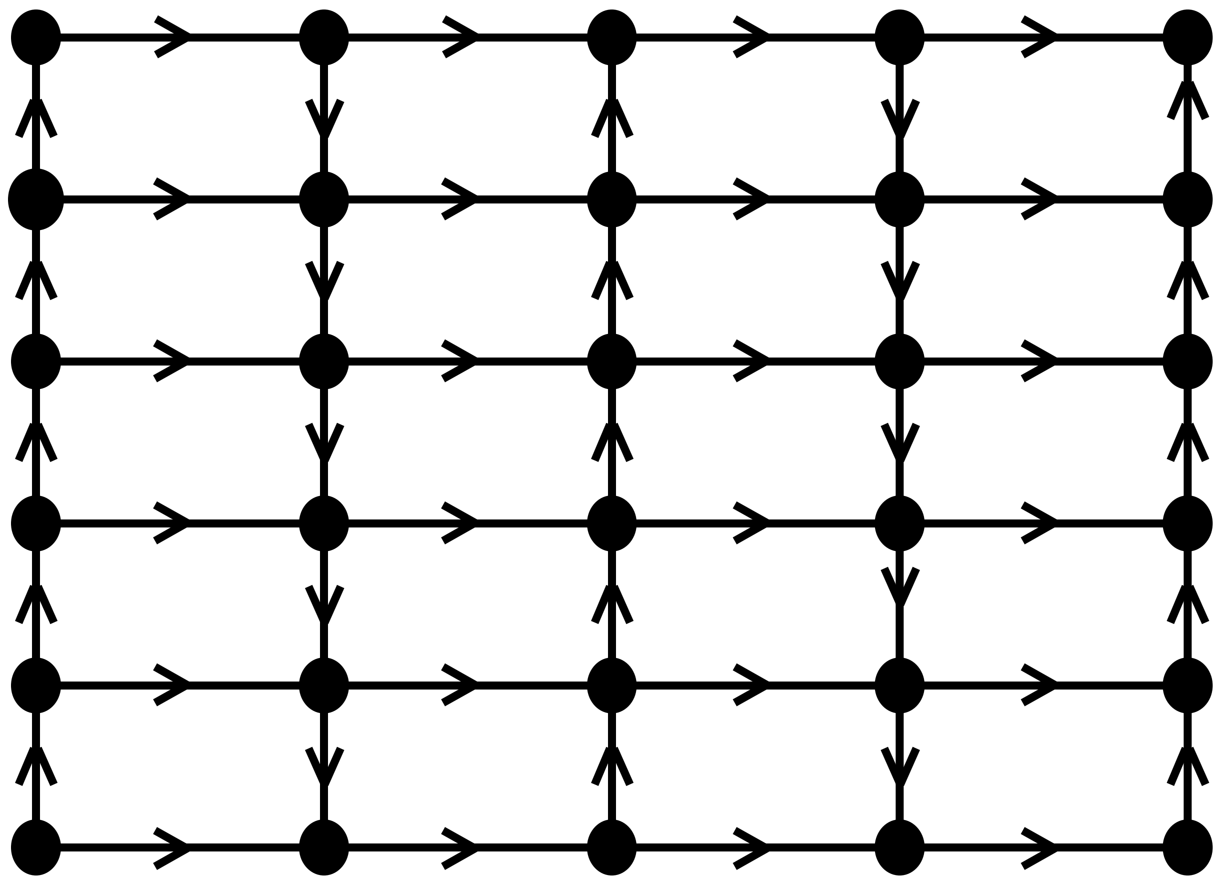
Equivalently: Every even cycle C such that $G \backslash V(C)$ has a perfect matching ("central cycle") has an odd number of edges directed in either direction (is "oddly oriented").

Equivalently: Either G has no perfect matching, or for some perfect matching M, every M-alternating cycle is oddly oriented.

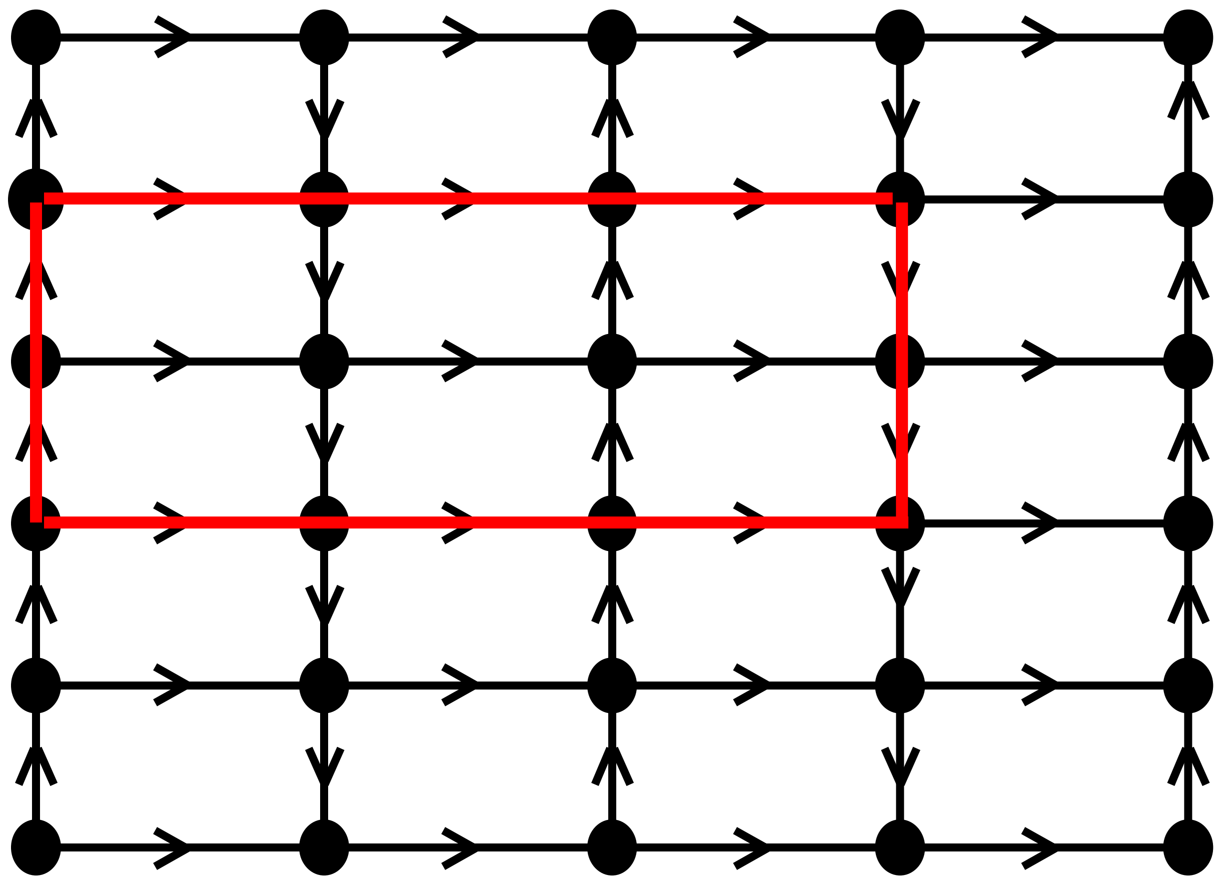
Example:



Example:

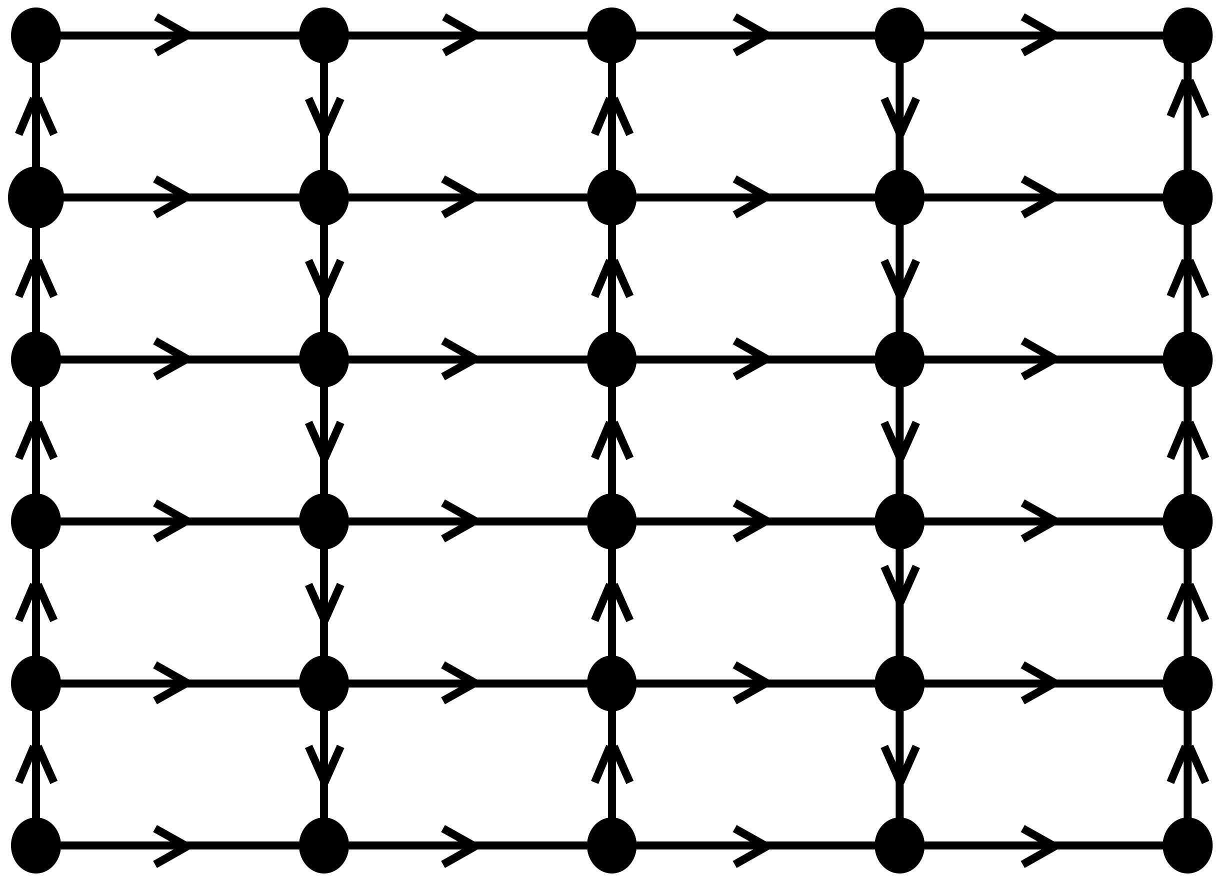


Example:

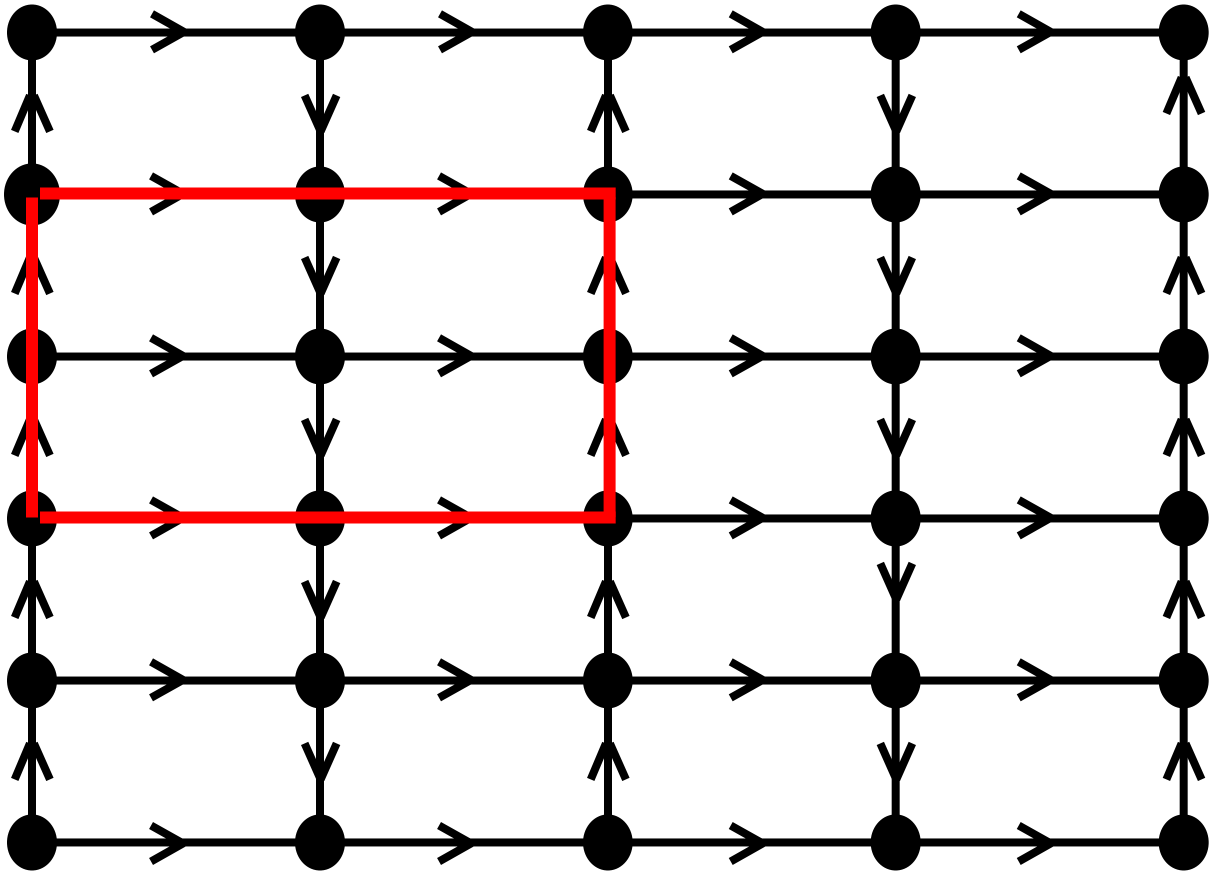


Oddly oriented

Example:

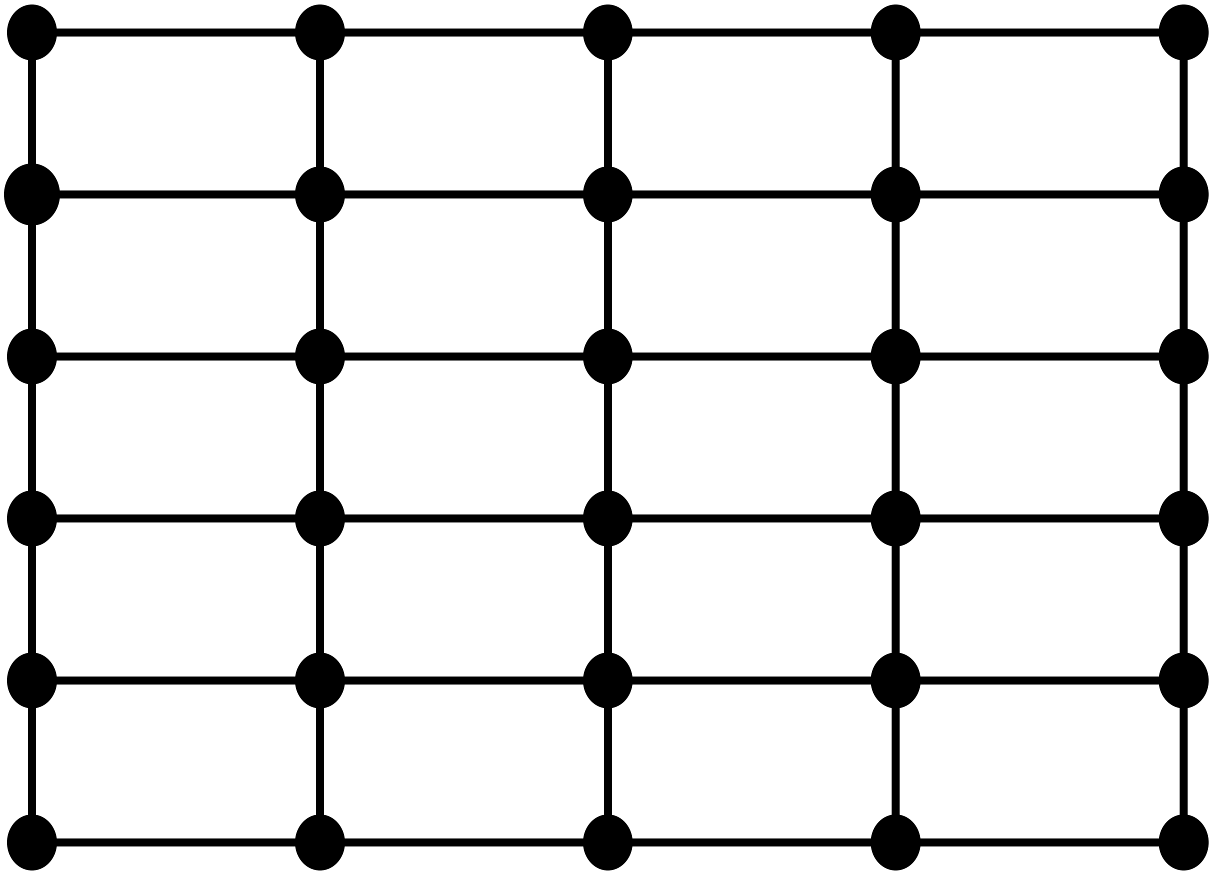


Example:



Not oddly oriented

Origins: Kasteleyn, Fisher, Temperley & Fisher



The number of perfect matchings is $\eta^{N(1+o(1))}$.

THM (Kasteleyn 1963) Every planar graph has a Pfaffian orientation.

PROOF Orient G so that $\forall$ cycle C:
C is clockwise odd $\Leftrightarrow$ C encloses even number of vertices of G.

Six equivalent problems

Let $A=(a_{i,j})_{i\,j=1,..,n}$ be a 0,1-matrix.

$$\det(A)= \sum \ \mathrm{sgn}(\sigma)\ a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma\,(n)}$$

$$\mathrm{per}(A)= \sum \ a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma\,(n)}$$

PROBLEM 1(Polya 1913) Given a square *0,1*-matrix *A*, does there exist a *0,1,-1*-matrix *B* obtained from *A* by changing some of the the *1*'s to *-1*'s in such a way that

per A = det B?

PROBLEM 1(Polya 1913) Given a square *0,1*-matrix *A*, does there exist a *0,1,-1*-matrix *B* obtained from *A* by changing some of the *1*'s to *-1*'s in such a way that

$$per\ A = det\ B?$$

EXAMPLE Not true for $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

PROOF

$$\sum_{\sigma} \mathrm{sgn}(\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)}$$

PROBLEM 1(Polya 1913) Given a square *0,1*-matrix *A*, does there exist a *0,1,-1*-matrix *B* obtained from *A* by changing some of the *1*'s to *-1*'s in such a way that

$$per\ A = det\ B?$$

EXAMPLE Not true for $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

PROOF

$$\prod_{\sigma} \mathrm{sgn}(\sigma)\ b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)}$$

PROBLEM 1(Polya 1913) Given a square *0,1*-matrix *A*, does there exist a *0,1,-1*-matrix *B* obtained from *A* by changing some of the *1*'s to *-1*'s in such a way that

$$per\ A = det\ B?$$

EXAMPLE Not true for $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

PROOF

$$1=\prod_{\sigma} \mathrm{sgn}(\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)}$$

PROBLEM 1 (Polya 1913) Given a square *0,1*-matrix *A*, does there exist a *0,1,-1*-matrix *B* obtained from *A* by changing some of the *1*'s to *-1*'s in such a way that

$$per\ A = det\ B?$$

EXAMPLE Not true for $J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

PROOF

$$1 = \prod_{\sigma} \operatorname{sgn}(\sigma)\, b_{1\sigma(1)} b_{2\sigma(2)} b_{3\sigma(3)} = (-1)^3\, b_{11}^2 b_{12}^2 b_{13}^2 \cdots b_{33}^2 = -1$$

PROBLEM 2 Given a bipartite graph, does it have a Pfaffian orientation?

PROBLEM 3 Given a directed graph, does it have no even directed cycle?

PROBLEM 3' Given a directed graph, is there a function w:$E(D)$->**Z** such that no directed cycle has even total weight?

A real $n \times n$ matrix A is sign-nonsingular if every real $n \times n$ matrix B with the same sign pattern is nonsingular.

PROBLEM 4. Given a square matrix, is it sign-nonsingular?

Application to sign-solvability. Given $A\mathbf{x}=\mathbf{b}$, is the sign pattern of $\mathbf{x}$ uniquely determined by the sign patterns of A and $\mathbf{x}$?

AN APPLICATION

Economic model of a banana trade:

S supply of bananas   *D* demand for bananas

p unit price of bananas   *t* people's taste for bananas

Then

(1) $\frac{\partial S}{\partial p} > 0, \quad \frac{\partial D}{\partial p} < 0, \quad \frac{\partial D}{\partial t} > 0$

Equilibrium equations and (1) imply that as people's taste for bananas increases, so do the price and supply(=demand). The general question leads to sign-solvability.

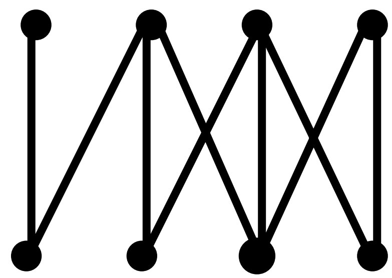
THEOREM. It is NP-hard to decide if a hypergraph is bipartite. It is NP-hard to decide if a hypergraph is minimally non-bipartite.

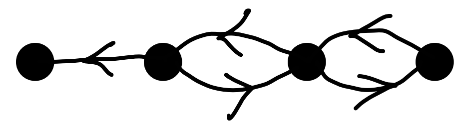
THEOREM (Seymour) If a hypergraph is minimally non-bipartite, then $|E| \geq |V|$.

PROBLEM 5. Given a hypergraph (V,E) with $|V|=|E|$ is it minimally non-bipartite?

Matrices to bipartite graphs to digraphs

1	0	0	0
1	1	1	0
0	1	1	1
0	0	1	1





Polya matrix iff Pfaffian orientation iff ∃ *w*:*E*(*D*)->**Z**

no even cycle

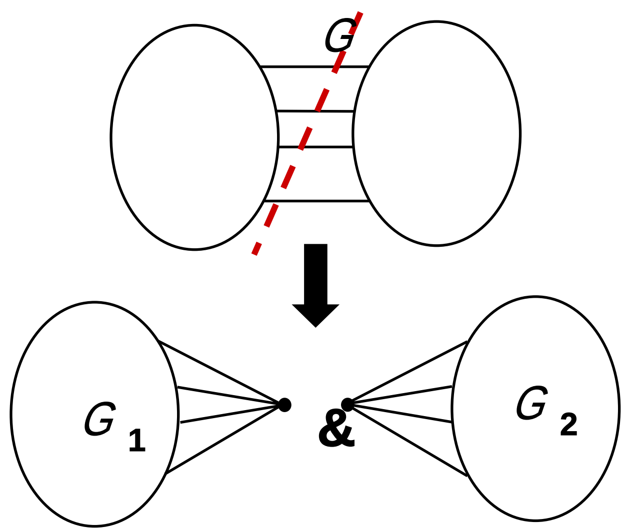
Characterizing bipartite Pfaffian graphs

THEOREM (Little 1975) A bipartite graph G has a Pfaffian orientation $\Leftrightarrow$ G has no $K_{3,3}$ matching minor.

G is a matching minor of H if G can be obtained from a central subgraph of H by bicontracting.

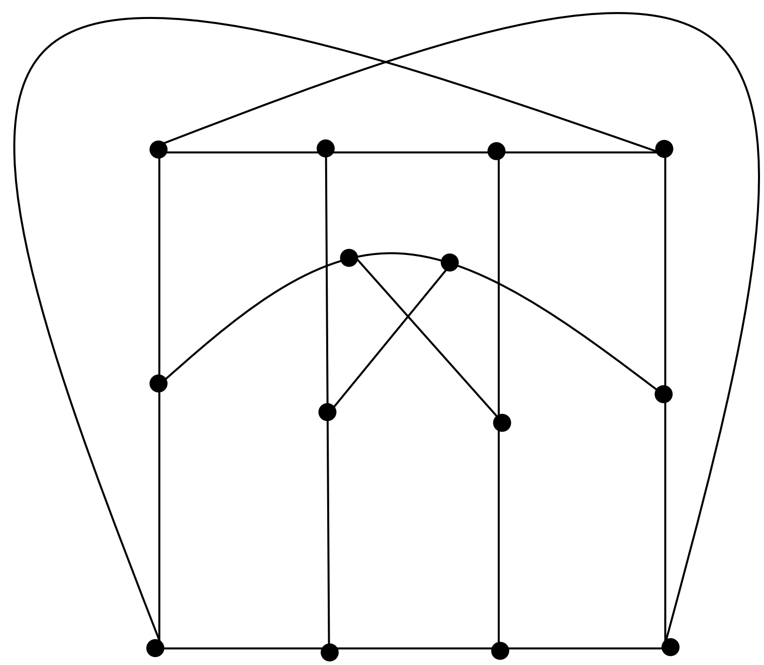
WMA every edge belongs to a perfect matching

A cut C in G is tight if $|C \cap M| = 1$ $\forall$ perfect matching M.
Tight cut decomposition:

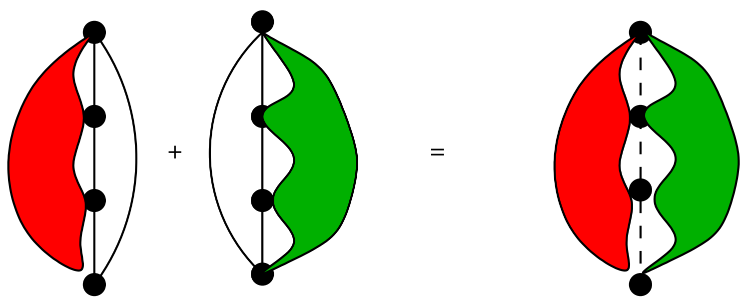


Bipartite graphs with no tight cut are braces, nonbipartite are called bricks.

The Heawood graph:



THEOREM (McCuaig; Robertson,Seymour,RT) A brace has a Pfaffian orientation ⇔ it either is isomorphic to the Heawood graph, or can be obtained by repeatedly C_4-summing, starting from planar braces.



COROLLARY. There is an $O(n^2)$ algorithm to solve the six problems mentioned earlier.

Pfaffian orientations in general graphs

Drawing Pfaffian Graphs

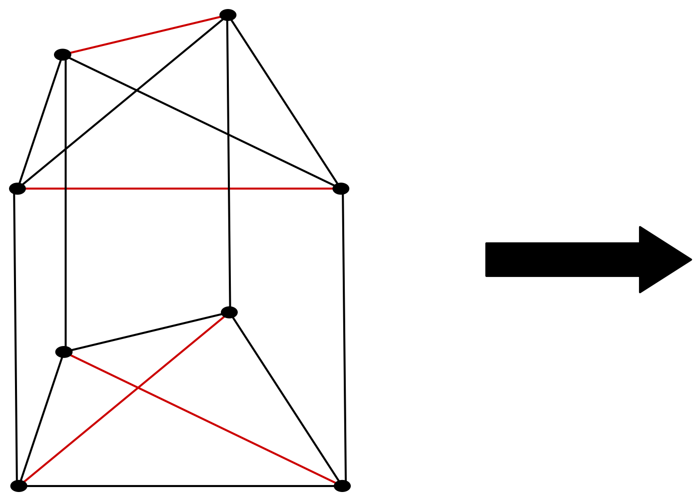
Theorem (Kasteleyn): Every planar graph is Pfaffian.

Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.

Drawing Pfaffian Graphs

Theorem (Kasteleyn): Every planar graph is Pfaffian.

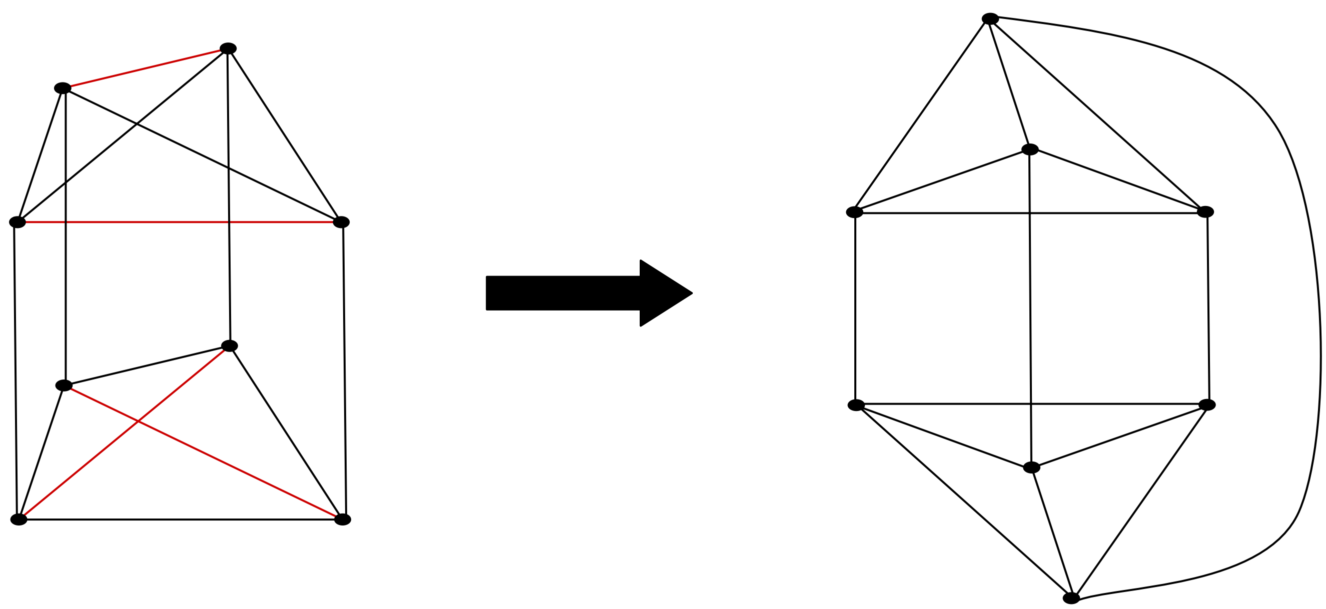
Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.



Drawing Pfaffian Graphs

Theorem (Kasteleyn): Every planar graph is Pfaffian.

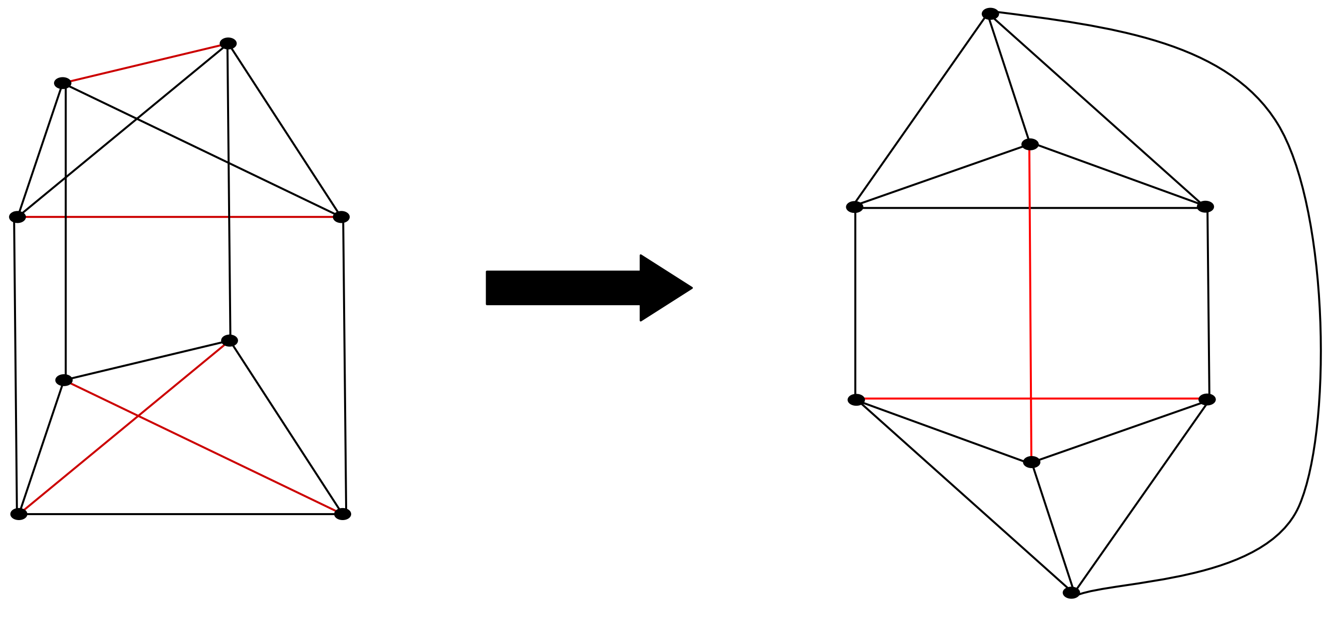
Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.



Drawing Pfaffian Graphs

Theorem (Kasteleyn): Every planar graph is Pfaffian.

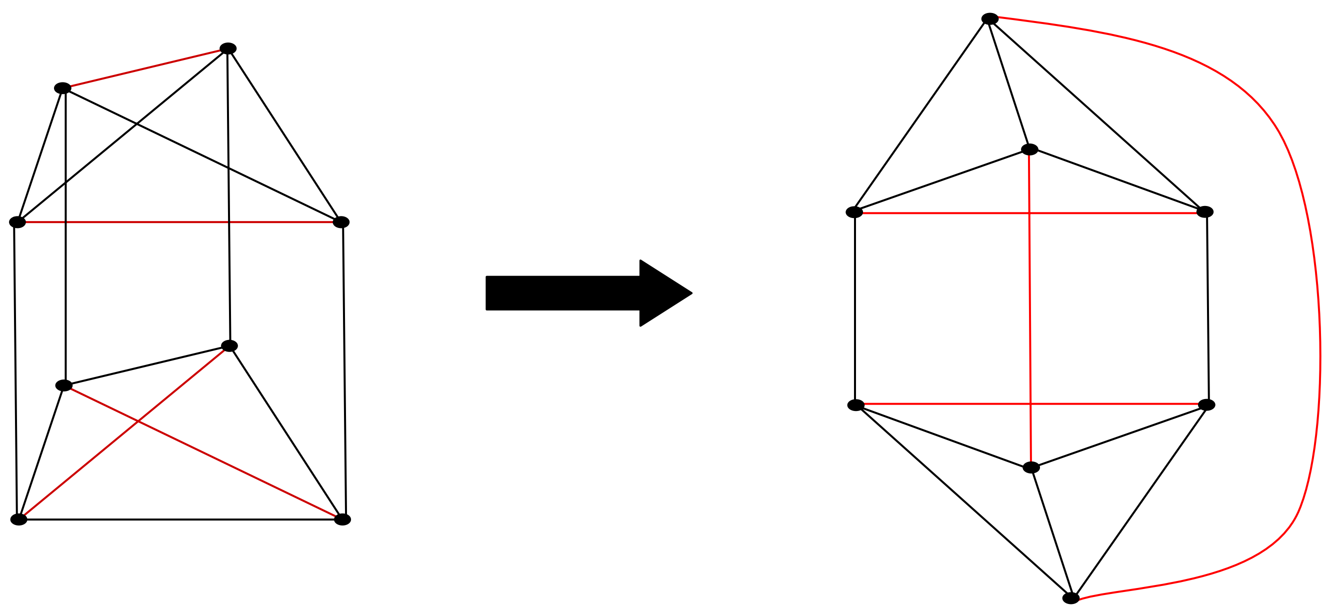
Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.



Drawing Pfaffian Graphs

Theorem (Kasteleyn): Every planar graph is Pfaffian.

Theorem (Norine) A graph is Pfaffian if and only if it can be drawn in the plane (possibly with crossings) so that every perfect matching intersects itself an even number of times.



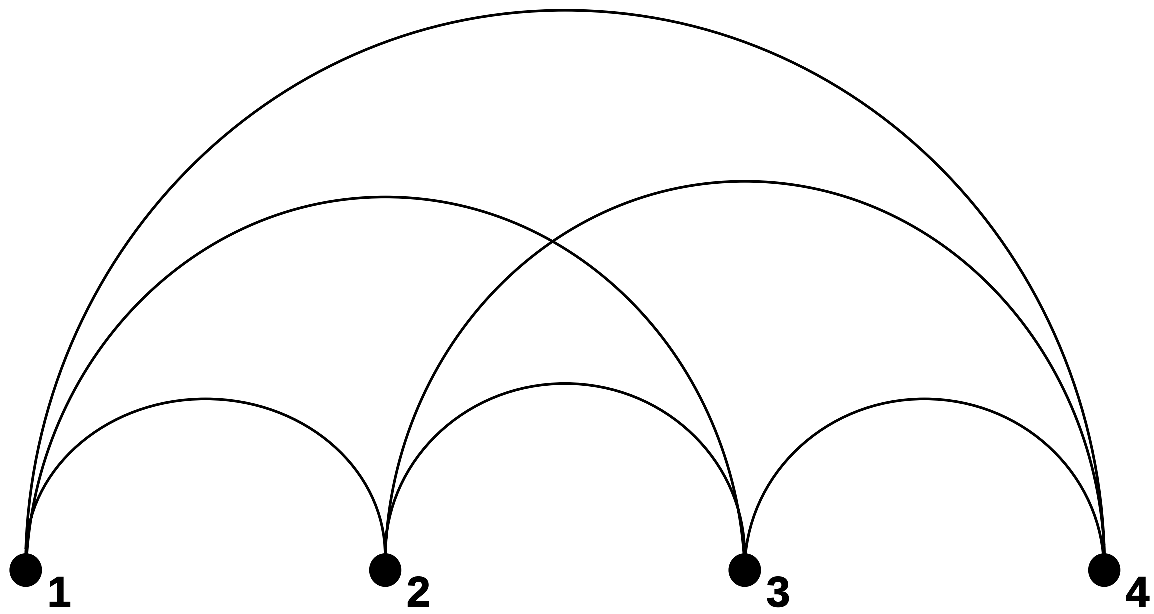
Proof:

$S \subseteq E(G)$ is a Pfaffian marking of a drawing of G in the plane if for every perfect matching M of G the parity of self intersections of M is equal to the parity of $|M \cap S|$.

Theorem: For a graph G, the following are equivalent:

1. G is Pfaffian
2. Some drawing of G in the plane has a Pfaffian marking
3. Every drawing of G in the plane has a Pfaffian marking
4. There exists a drawing of G in the plane such that every perfect matching intersects itself even number of times.

Standard Drawing

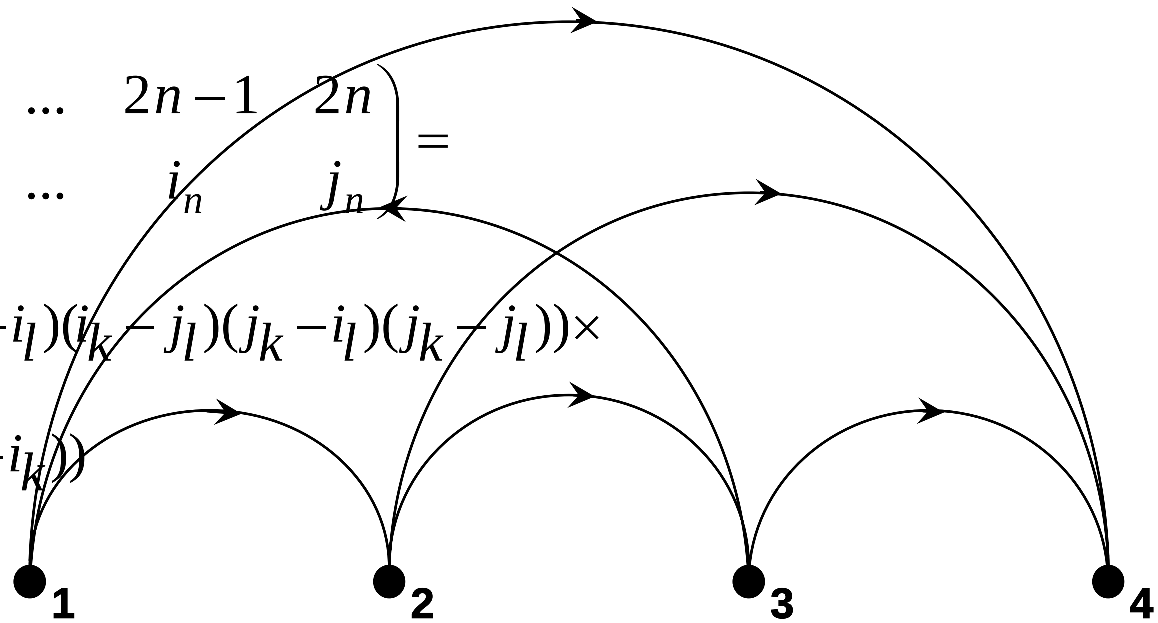


Standard Drawing

$$sign\begin{pmatrix}1 & 2 & \dots & 2n-1 & 2n\\ i_1 & j_1 & \dots & i_n & j_n\end{pmatrix} =$$

$$= sign\left(\prod_{k>l}(i_k-i_l)(i_k-j_l)(j_k-i_l)(j_k-j_l)\right)\times$$

$$\times sign\left(\prod_{k}(j_k-i_k)\right)$$

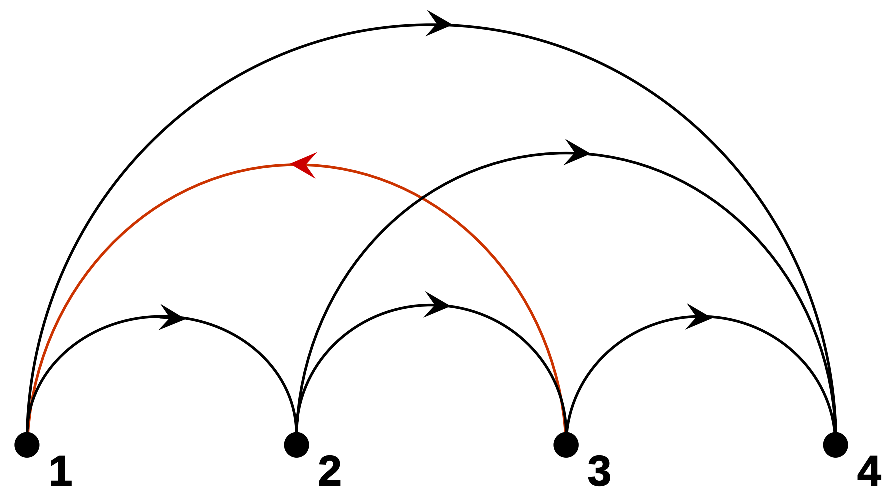


$$sign((i_k-i_l)(i_k-j_l)(j_k-i_l)(j_k-j_l)) = -1$$

if and only if edges *k* and *l* intersect.

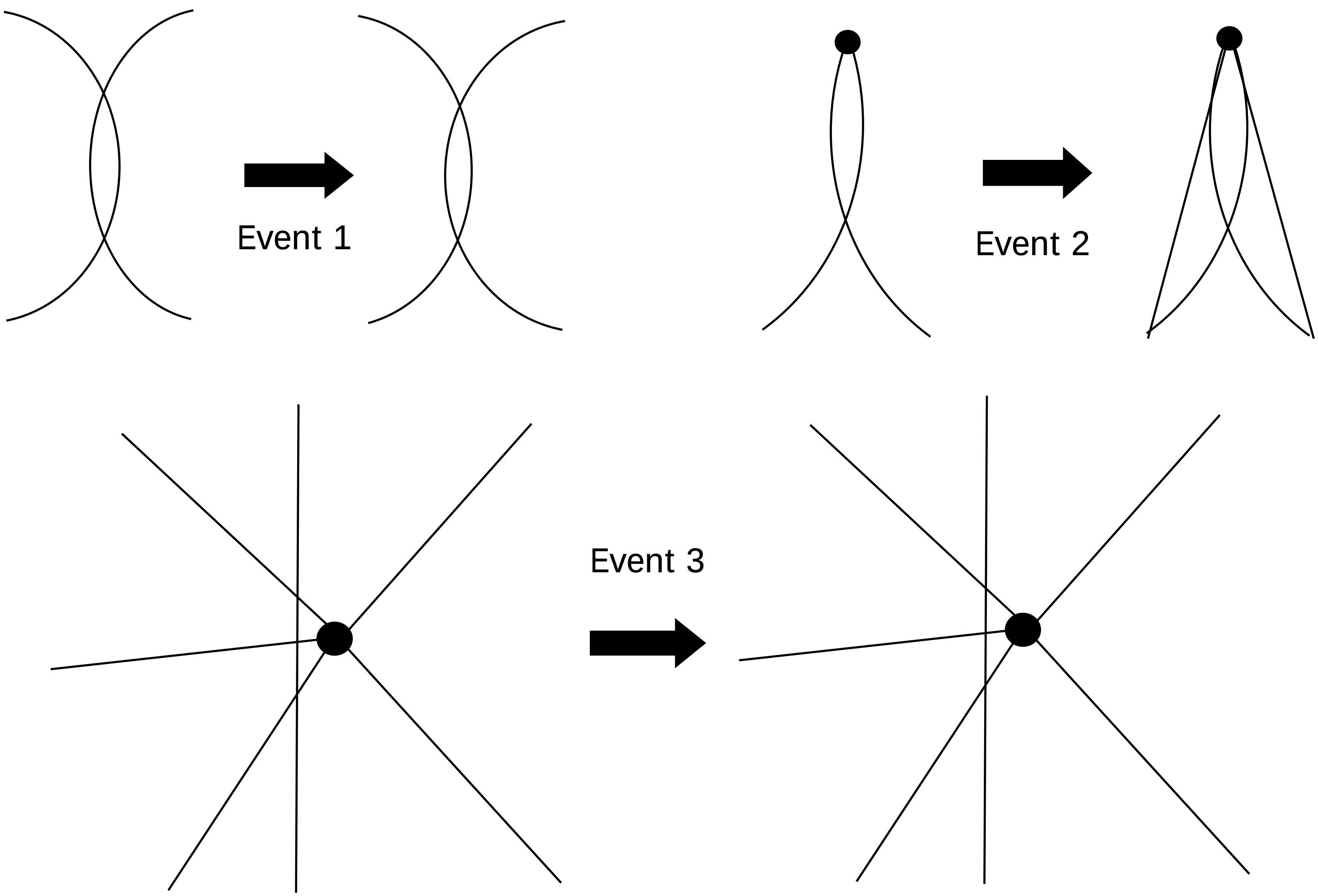
Standard Drawing

In a standard drawing of a graph with a Pfaffian orientation, the set of backward edges is a Pfaffian marking.

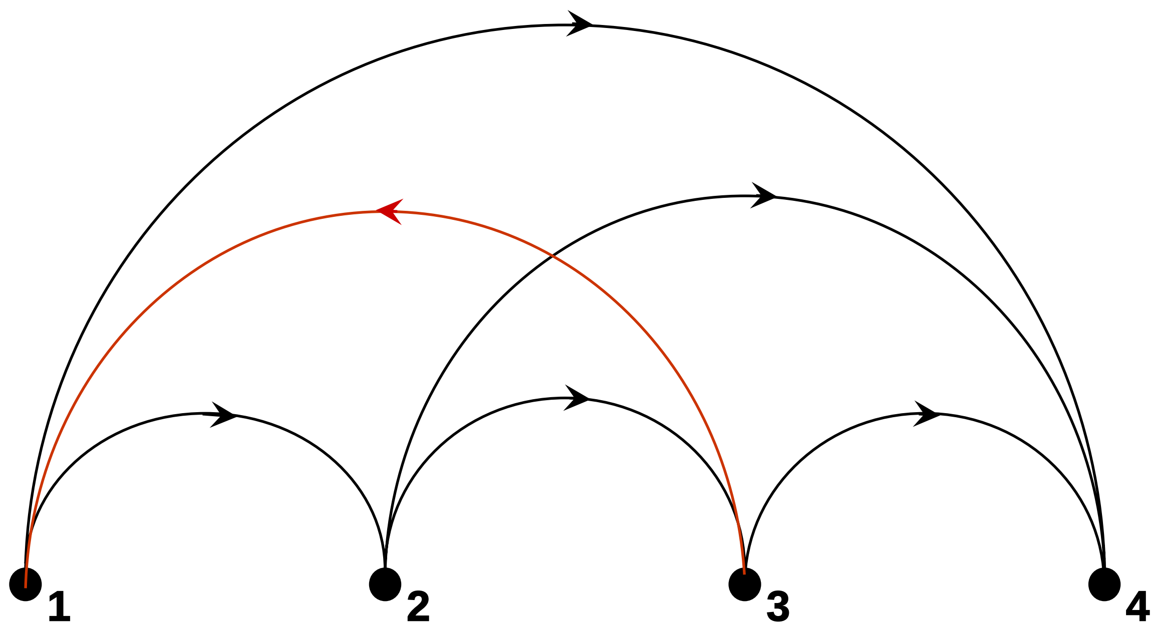


If *S* is a Pfaffian marking of a standard drawing of a graph then the orientation in which backward edges are exactly the edges of *S* is Pfaffian.

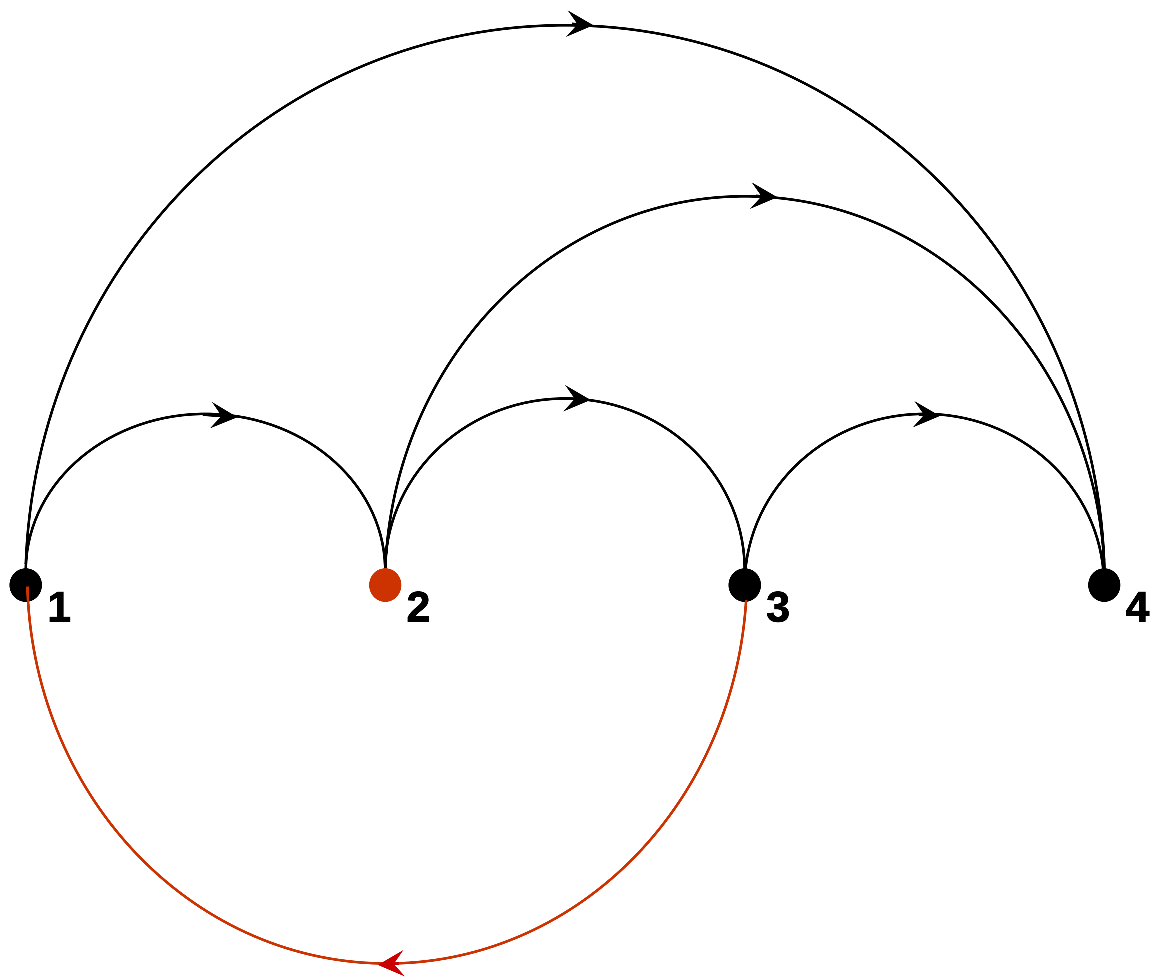
Changing The Drawing



Changing The Drawing



Changing The Drawing



T-joins and Crossing Numbers

Norine proved a more general theorem about T-joins. Corollaries:

•**Theorem** (**Hannani, Tutte**) If G can be drawn in the plane in such a way that every two edges cross even number of times, then G is planar.

•**Theorem** (**Kleitman**): Let $G=K_{2j+1}$ or $G=K_{2j+1,2k+1}$. Then the parity of the total number of crossings of non-adjacent edges is independent of the choice of the drawing of G in the plane.

• Purely combinatorial reformulation of Turan's brickyard problem (the problem of estimating the crossing number of a complete bipartite graph).

A labeled graph G is k-Pfaffian if there exist orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every perfect matching M of G

$$\sum_{i=1}^{k} \alpha_i \, \mathrm{sgn}_{D_i}(M) = 1.$$

Theorem(Gallucio, Loebl; Tesler, 1999): Every graph that can be embedded in the surface of genus g is 4^g- Pfaffian.

Theorem (Norine) Every 3-Pfaffian graph is Pfaffian.
Theorem (Norine) A graph is 4-Pfaffian if and only if it can be drawn on the torus (possibly with crossings) so that every perfect matching intersects itself an even number of times.
Theorem (Norine) Every 5-Pfaffian graph is 4-Pfaffian.

A labeled graph G is k-Pfaffian if there exist orientations $D_1, D_2, \ldots, D_k$ of G and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every perfect matching M of G

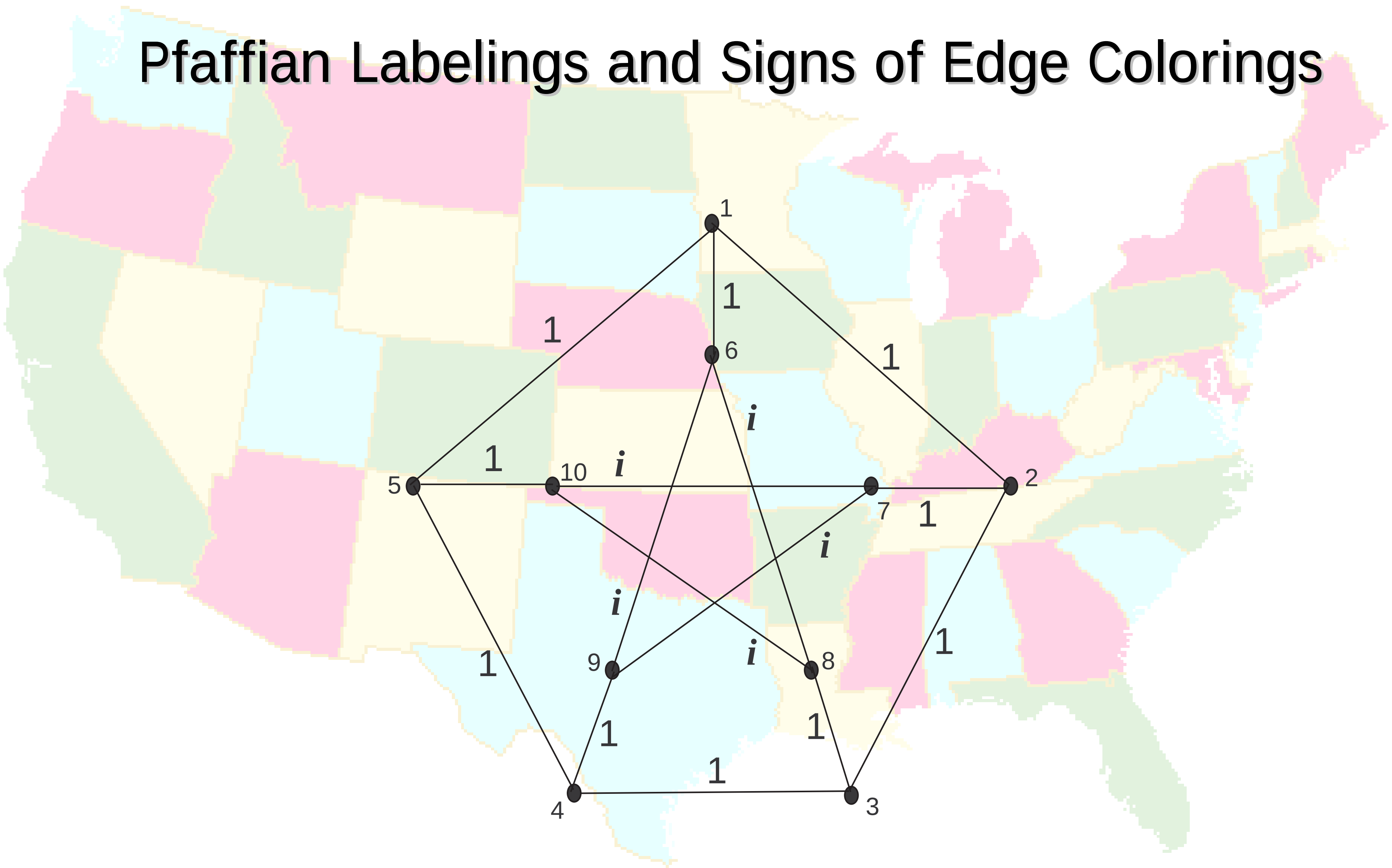
$$\sum_{i=1}^{k} \alpha_i \operatorname{sgn}_{D_i}(M) = 1.$$

Theorem (**Gallucio**, **Loebl**; **Tesler**, **1999**)**:** Every graph that can be embedded in the surface of genus g is 4^g- Pfaffian.

Conjecture: For a graph G and integer $g \geq 0$ TFAE:

1. There exists a drawing of G on an orientable surface of genus g such that every perfect matching intersects itself an even number of times.
2. G is 4^g -Pfaffian.
3. G is $(4^{g+1}-1)$-Pfaffian.

Pfaffian Labelings and Signs of Edge Colorings



A graph G is k-edge-choosable if for every set system $\{S_e: e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge-coloring c with $c(e) \in S_e$ for every $e \in E(G)$.

List Edge Coloring Conjecture: Every k-edge colorable graph is k-edge-choosable.

THM (Ellingham,Goddyn, based on Alon,Tarsi)
True for k-regular planar graphs

THM (Norine,RT, based on Alon,Tarsi)
True for k-regular Pfaffian graphs

The proof uses “signs” of edge-colorings, and in a sense the method works only for Pfaffian graphs

A graph G is k-edge-choosable if for every set system $\{S_e: e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge-coloring c with $c(e) \in S_e$ for every $e \in E(G)$.

List Edge Coloring Conjecture: Every k-edge colorable graph is k-edge-choosable.

THM (Ellingham,Goddyn, based on Alon,Tarsi)
True for k-regular planar graphs

THM (Norine,RT, based on Alon,Tarsi)
True for k-regular Pfaffian graphs

CONJECTURE Every 2-connected 3-regular Pfaffian graph is 3-edge-colorable

A graph G is k-edge-choosable if for every set system $\{S_e: e \in E(G)\}$ such that $|S_e| = k$ there exists a proper edge-coloring c with $c(e) \in S_e$ for every $e \in E(G)$.

List Edge Coloring Conjecture: Every k-edge colorable graph is k-edge-choosable.

THM (Ellingham,Goddyn, based on Alon,Tarsi)
True for k-regular planar graphs

THM (Norine,RT, based on Alon,Tarsi)
True for k-regular Pfaffian graphs

CONJECTURE Every 2-connected 3-regular Pfaffian graph is 3-edge-colorable (Implies the 4-color theorem)

Pfaffian Labelings

Let Γ be an Abelian group, we assume $1,-1 \in \Gamma$.

Let G be a graph. $V(G)=\{1,2, \ldots, 2n\}$.

$l: E(G) \to \Gamma$ is a Pfaffian labeling of G if for every perfect matching

$$M=\{\{ i_1 , j_1 \}, \{ i_2 , j_2 \}, \ldots, \{ i_n , j_n \}\}, \; i_k<j_k$$

we have

$$\prod_{e \in M} l(e) = sign\begin{pmatrix} 1 & 2 & \ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots & i_n & j_n \end{pmatrix}.$$

The previous theorem holds more generally for graphs that admit a Pfaffian labeling, but those are not much different from Pfaffian graphs

THEOREM (Little 1975) A bipartite graph G has a Pfaffian orientation $\Leftrightarrow$ G has no $K_{3,3}$ matching minor.

G is a matching minor of H if G can be obtained from a central subgraph of H by bicontracting.

THEOREM (Fischer, Little) A near-bipartite graph G has a Pfaffian orientation $\Leftrightarrow$ G has no matching minor isomorphic to $K_{3,3}$, Γ_1, or Γ_2.

For general graphs need to add infinitely many graphs.

Structure of Pfaffian graphs

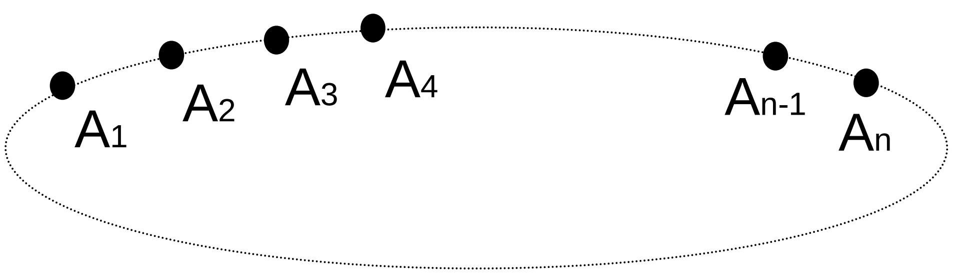
Basic classes of Pfaffian graphs:

- Planar graphs
- Graphs which have “even-faced” embeddings in the Klein bottle

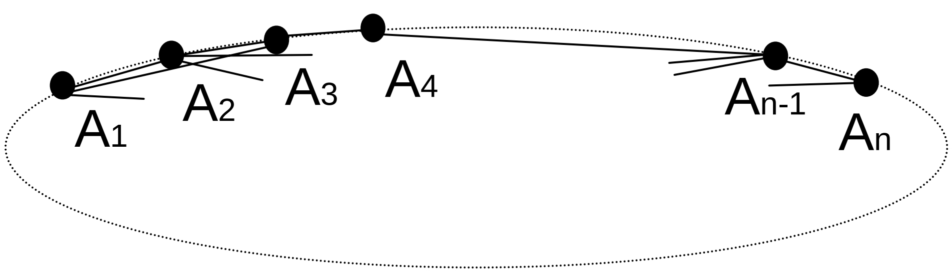
Is there a decomposition theorem?

Obstacle: dense Pfaffian bricks

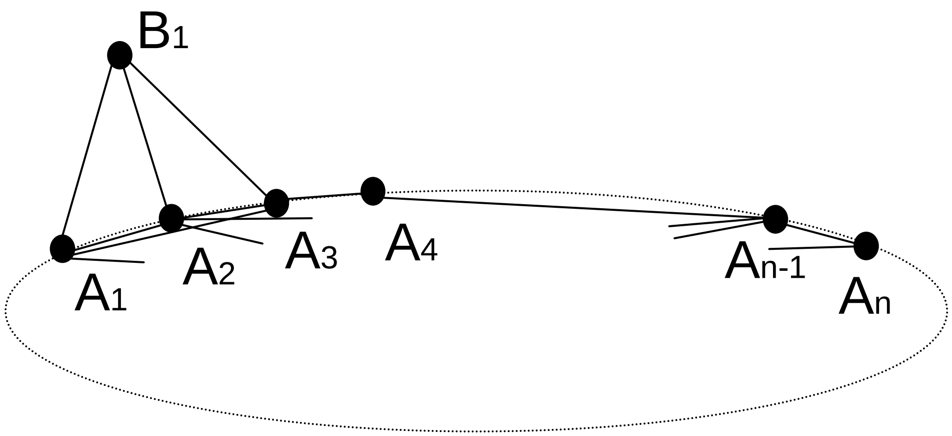
Dense Pfaffian Bricks



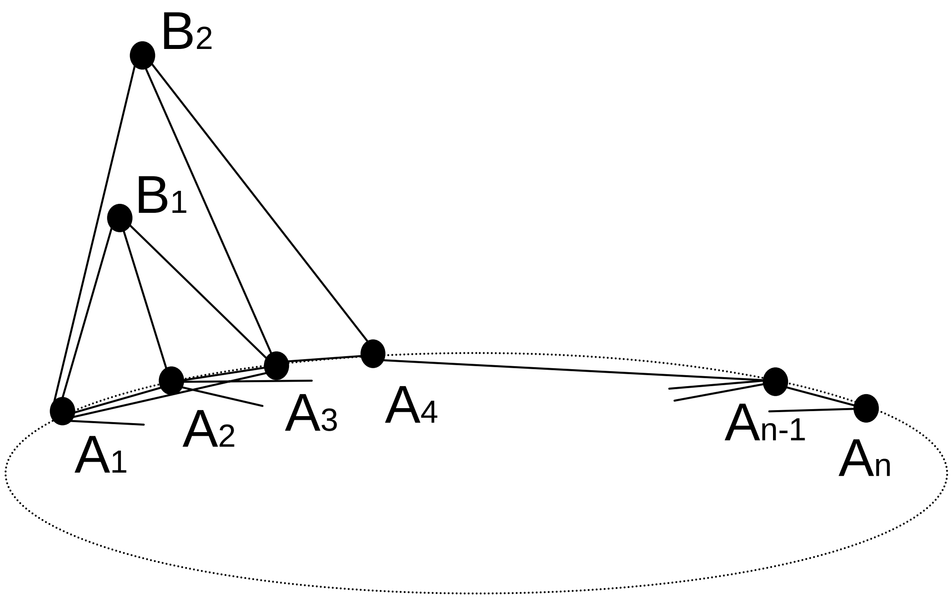
Dense Pfaffian Bricks



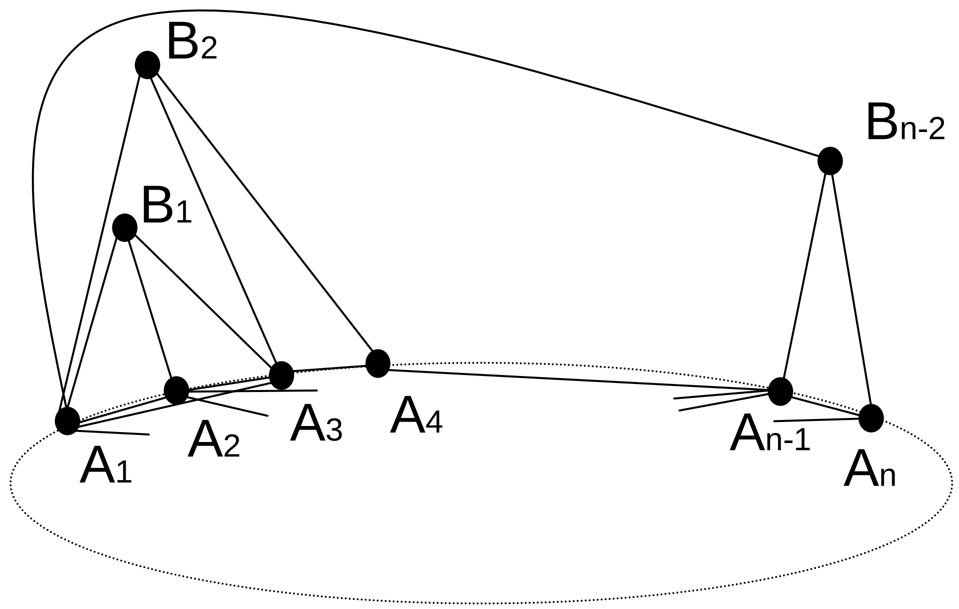
Dense Pfaffian Bricks



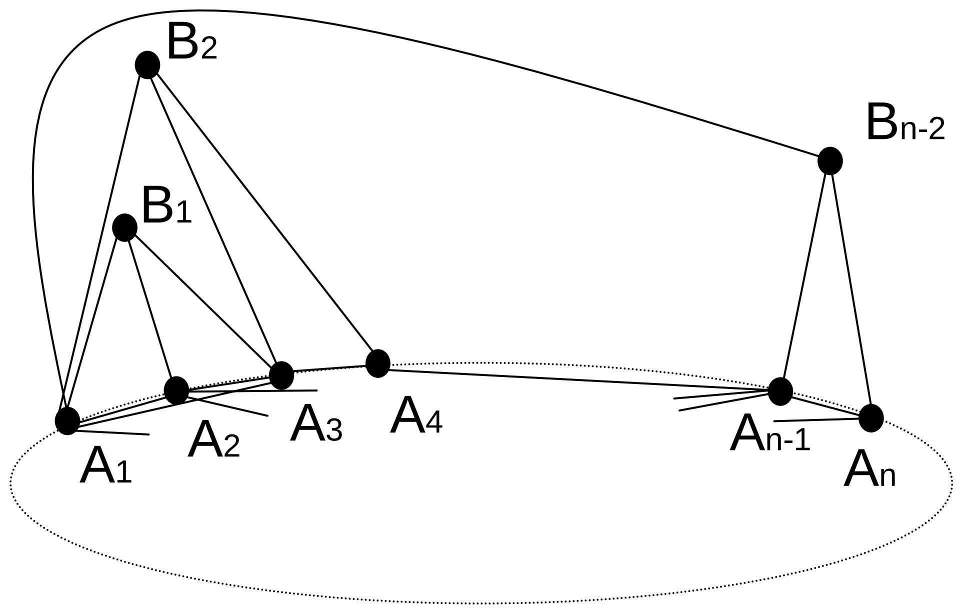
Dense Pfaffian Bricks



Dense Pfaffian Bricks



Dense Pfaffian Bricks

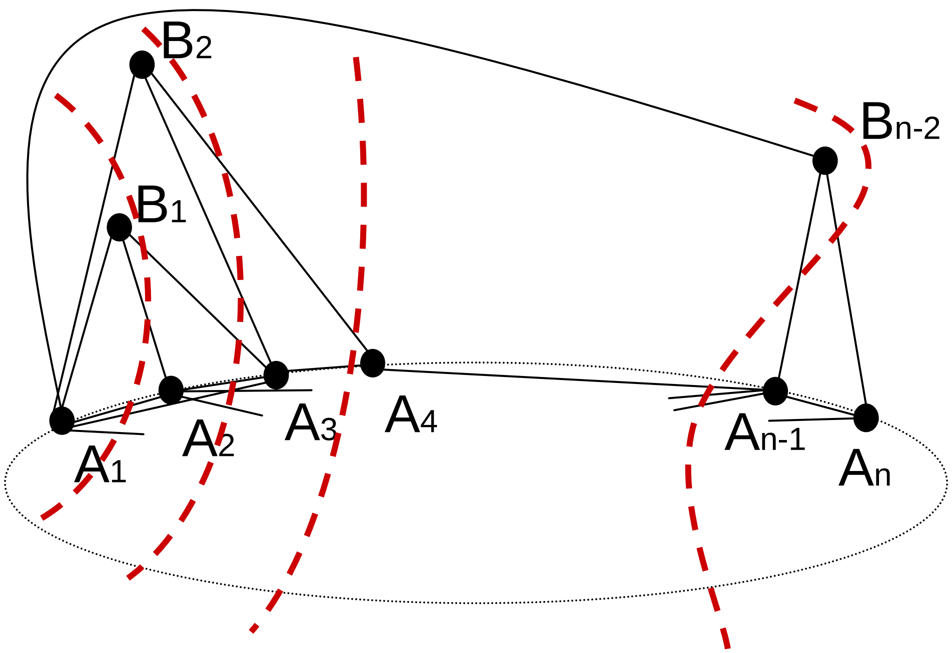


$2n-2$ vertices

$(n^2 + 5n - 12)/2$ edges

K_n subgraph

The tightness of a cut C in a graph G is the maximum of $|M \cap C|$ over all perfect matchings M of G.



Tightness leads to the notion of matching-width, analogous to tree-width.

THM (Norine, RT) Can test in poly time if a graph of bounded matching-width is Pfaffian

CONJECTURE Huge matching-width $\Rightarrow$ large grid matching minor

SUMMARY

Bipartite Pfaffian graphs are well-understood, and their characterization solves other problems

General Pfaffian graphs are not, but there are interesting connections to other areas