LINKLESS EMBEDDINGS OF GRAPHS IN 3-SPACE

Graphs are finite, undirected, may have loops and multiple edges.
All embeddings are piecewise linear

KNOT THEORY

THEOREM (Papakyriakopolous) A simple closed curve in $\mathbb{R}^3$ is unknotted $\Leftrightarrow$ its complement has free fundamental group.

Linking number

$lk(C, C') =$

An embedding of G in 3-space is linkless if $lk(C, C') = 0$ for every two disjoint circuits C, C' of G.

THEOREM (Conway and Gordon, Sachs) K_6 does not have a linkless embedding.

PROOF Fix embedding of K_6. Consider

$$\sum lk(C, C') \bmod 2,$$

summing over all unordered pairs of disjoint circuits C, C' of G.

- Does not depend on the embedding
- $= 1$ for some embedding

The Petersen family is the set of 7 graphs that can be obtained from K_6 by repeatedly applying $Y\Delta$- and ΔY-operations

THEOREM (Sachs) No member of the Petersen family has a linkless embedding.

An embedding of a graph G in 3-space is flat if every circuit of G bounds a disk disjoint from the rest of G.

spherical $\Rightarrow$ flat $\Rightarrow$ linkless

MAIN THEOREM

A graph has a flat embedding $\Leftrightarrow$ it has no minor isomorphic to a member of the Petersen family.

COROLLARY $\Leftrightarrow$ it has a linkless embedding

An embedding of a graph in $\mathbb{R}^3$ is spherical if the graph lies on a surface homeomorphic to S^2.

THEOREM (Scharlemann, Thompson) An embedding of a graph G is spherical $\Leftrightarrow$

(i) G is planar, and

(ii) $\pi_1(\mathbb{R}^3 - G')$ is free for every subgraph G' of G

THEOREM An embedding of a graph G is flat $\Leftrightarrow \pi_1(\mathbb{R}^3 - G')$ is free for every subgraph G' of G.

COROLLARY If G is embedded non-flatly then for every non-loop edge e either $G\backslash e$ or G/e is not flat.

THEOREM (Böhme, Saran) Let G be a flatly embedded graph, let $C_1, C_2, \ldots, C_n$ be circuits with pairwise connected intersection. Then there are disjoint open disks $D_1, D_2, \ldots, D_n$, disjoint from the graph and s.t. $\partial D_i = C_i$.

COROLLARY If G is planar and $\phi : G \to \mathbb{R}^3$ is an embedding, then ϕ is flat if and only if ϕ is spherical ($= \phi(G)$ lies on an embedded sphere).

PLANAR EMBEDDINGS

WHITNEY'S THEOREM Any two planar embeddings of the same graph differ (up to isotopy) by 2-switches.

KURATOWSKI'S THEOREM A graph is planar $\Leftrightarrow$ it has no K_5 or $K_{3,3}$ minor.

LEMMA In a 4-connected graph G, every two $K_{3,3}$ minors "communicate".

THEOREM Two nonisotopic flat embeddings differ on a $K_{3,3}$ or K_5 minor.

PROOF Contract an edge. WMA now isotopic.
Different equators:

UNIQUENESS THEOREM Let ϕ_1, ϕ_2 be two flat embeddings of a 4-connected graph G. Then either $\phi_1 \simeq \phi_2$ or $\phi_1 \simeq -\phi_2$.

ABOUT THE PROOF WMA G has a $K_{3,3}$-minor, call it H. WMA $\phi_1 \restriction H \simeq \phi_2 \restriction H$ (by replacing ϕ_2 by $-\phi_2$ if necessary).

By the Lemma, $\phi_1 \restriction H' \simeq \phi_2 \restriction H'$ for every $K_{3,3}$-minor H. By the previous theorem, $\phi_1 \simeq \phi_2$.

FACT Every planar graph has a unique flat embedding.

PROPOSITION $K_{3,3}$ and K_5 have two flat embeddings (up to isotopy).

COROLLARY A graph has a unique flat embedding $\Leftrightarrow$ it is planar.

OUTLINE OF PROOF Take a minor-minimal counterexample, G, WMA no triangles. It can be shown G is "internally 5-connected." Take edges $e = uv$, f such that $G/f/e$, $G/f\backslash e$ are "Kuratowski connected."
Let ϕ_1 be a flat embedding of $G\backslash e$.
Let ϕ_2 be a flat embedding of G/e.
Let ϕ_3 be a flat embedding of G/f. WMA

$$\phi_1/f \simeq \phi_3\backslash e$$
$$\phi_2/f \simeq \phi_3/e$$

It can be shown that the uncontraction of f is the same in both of these embeddings. Let ϕ be obtained from ϕ_3 by doing this uncontraction. Then $\phi\backslash e \simeq \phi_1$, $\phi/e \simeq \phi_2$, and so ϕ is flat.