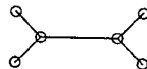


5. [BB; A only] If we start at A , permanent labels will be assigned in the order $A, H, B, C, D, J, I, G, F, K, E$. If we start at H , permanent labels will be assigned in the order H, A, B, C, D, I or J (at same time), G, F, K, E .
17. (a) [BB] The final values are the shortest distances from A to itself and to the other vertices. From Fig. 10.23, we deduce these to be 0, 7, 15, 20, 21, 15, 8, 5, 10 and 17.
- (b) [BB] After $k = 4$, the value of $d(1, 5)$ is the length of a shortest path from A to E via the vertices A, B, C, D . The only such path is $ABCDE$ of length 27: $d(1, 5) = 27$.
The value of $d(1, 6)$ is the length of a shortest path from A to F via A, B, C, D . There is no such path, so $d(1, 6)$ is still ∞ .
The value of $d(3, 4)$ is 8.
The value of $d(8, 5)$ is the length of a shortest path from H to E via A, B, C, D . The shortest such path is $HBCDE$: $d(8, 5) = 24$.
- (c) [BB] The initial value of $d(2, 5)$ is ∞ since BE is not an edge. After $k = 1, 2, \dots, 10$, the values of $d(2, 5)$ are $\infty, \infty, \infty, 20, 20, 20, 20, 20, 18, 17$.
27. (a) [BB] Using Corollary 12.1.7, we have $\sum \deg(v_i) \geq 8$, so the tree has at least four edges and hence at least five vertices. If the result is not true, then there are two vertices of degree 3, at most three vertices of degree 1, and the rest have degree at least 2. Then

$$\sum \deg v_i \geq 2(3) + 3(1) + (n - 5)2 = 2n - 1,$$

contradicting the fact that $\sum \deg v_i = 2(n - 1)$.

- (b) Here is a tree with two vertices of degree 3 and exactly four vertices of degree 1.



8. Let the bipartition sets be \mathcal{V} and \mathcal{W} and let $\mathcal{V} = \{v_1, v_2\}$ be the one of order two. In any spanning tree of $\mathcal{K}_{2,n}$, if two vertices of \mathcal{W} are adjacent to both v_1 and v_2 , there would be a cycle, which is not allowed. On the other hand, if no vertex of \mathcal{W} was adjacent (in a spanning tree) to both v_1 and v_2 , then the tree would not be connected, another contradiction. Thus, exactly one of the n vertices in \mathcal{W} is adjacent to both v_1 and v_2 . It follows that a spanning tree with k vertices joined to v_1 is obtained by selecting one of the n vertices of \mathcal{W} to be adjacent to both v_1 and v_2 , then (if $k > 1$) $k - 1$ of the remaining $n - 1$ vertices of \mathcal{W} to join to v_1 and finally joining the remaining $n - k$ vertices of \mathcal{W} to v_2 (no choice involved here). The number of spanning trees with $k > 1$ vertices joined to v_1 is $n \binom{n-1}{k-1}$.

When $k = 1$, there are n spanning trees. Thus, the total number of spanning trees is

$$n + n \sum_{k=2}^n \binom{n-1}{k-1} = n + n \sum_{k=1}^{n-1} \binom{n-1}{k} = n \left(1 + \sum_{k=0}^{n-1} \binom{n-1}{k} \right) = n \sum_{k=0}^{n-1} \binom{n-1}{k} = n 2^{n-1}.$$

12. (a) [BB] Say the edge is e and \mathcal{T} is any spanning tree. If e is not in \mathcal{T} , then $\mathcal{T} \cup \{e\}$ must contain a circuit. Deleting any edge of this circuit other than e gives another spanning tree which includes e .
- (b) Let e_1 and e_2 be the given edges and let \mathcal{T} be any spanning tree. By part (a), we may assume that \mathcal{T} contains e_1 . If it also contains e_2 , we are done; otherwise, $\mathcal{T} \cup \{e_2\}$ must contain a circuit \mathcal{C} to which e_2 belongs. Since circuits contain at least three edges, \mathcal{C} contains some edge g different from e_1 and different from e_2 . Now $(\mathcal{T} \cup \{e_2\}) \setminus \{g\}$ is a spanning tree containing both e_1 and e_2 .
- (c) [BB] No. If the three edges form a circuit, no spanning tree can contain them.
13. Let \mathcal{T} be a spanning tree for \mathcal{G} and let v be a vertex of degree 1 in \mathcal{T} . Since $\mathcal{T} \setminus \{v\}$ is connected, so is $\mathcal{G} \setminus \{v\}$. The final statement follows from the fact that any tree has at least two vertices of degree 1.