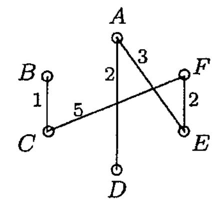
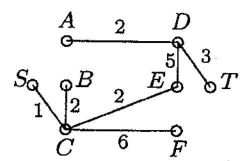
Hwk #11

2. (a) [BB] The edge of least weight incident with E is EF. The least weight of those edges adjacent to EF is 3; we choose one of them, say AE (in an effort to obtain a different tree from before). There is just one edge of least weight among whose which, together with EF and AE, form a tree; namely, AD. Now those edges which together with EF, AE, and AD form a tree have weights 5 and 6. We choose one of least weight, say CF. Finally, we choose BC, obtaining the tree of weight 13 shown above right.

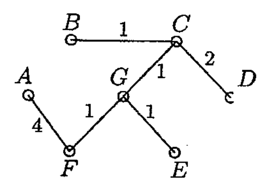


   (b) The edge of least weight incident with E is FE. The edge of least weight among those adjacent to FE is AE. The edge of least weight among those which, together with FE and AE, form a tree is AD. Continuing to form a sequence of ever-growing trees, we are forced to select CA and then BC. We obtain the same tree as before, as could have been predicted. (See Exercise 11.)

   (c) The edge of least weight incident with E is CE. The edge of least weight among those adjacent to CE is SC. The edge of least weight among those which, together with CE and SC, form a tree is BC. Then we must add, in order, DE, AD, and DT. Finally, so as not to duplicate the tree we obtained in Exercise 1, we select CF.



   (d) Starting at vertex E, we select first edge GE of weight 1 and then GF (to change the tree we had before). Of the edges incident with G, E and F, those of least weight are FE and GC. We cannot select FE because it is incident with **two** of the vertices already chosen, so we choose GC, then BC, CD and AF. We obtain another spanning tree of weight 10.

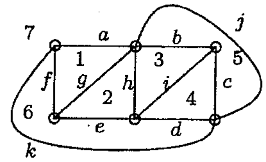


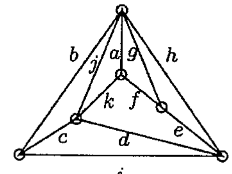
5. (a) Start with any vertex and the edge of maximum weight incident with that vertex. Then pick an edge of maximum weight from among those edges adjacent to the first. In general, given a subgraph of the given graph which is a tree but not a spanning tree, add to this an edge of maximum weight from those edges one of whose end vertices is not in the tree and the other of which is (so that the augmented tree is also a tree). Continue until there are no vertices outside the tree, that is, until the tree is a spanning tree.

   iii. Starting at vertex A, the first edge we must include is AE. Then we might select ET, FE, SA and then SB, CF and DE. We obtain the same tree as before.

Exercises 13.1

1. (a) [BB] We draw the graph quickly as a planar graph and then, after some thinking, as a plane with straight edges.

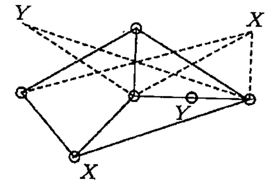




   (b) [BB] There are seven regions, numbered $1, 2, \ldots, 7$, with boundaries afg, ghe, hbi, icd, bjc, $fedk$, and ajk, respectively.

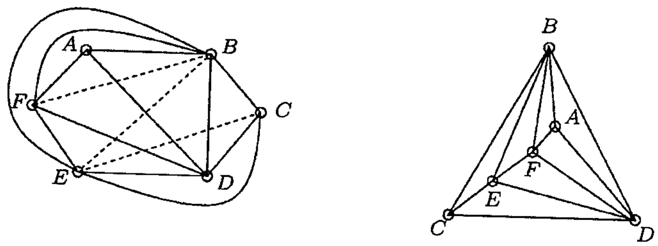
   (c) [BB] $E = 11$, $V = 6$, $R = 7$, $N = 22$; so $V - E + R = 6 - 11 + 7 = 2$; $N = 22 \leq 22 = 2E$ and $E = 11 \leq 12 = 3V - 6$.

5. (a) [BB] This is planar. Here it is drawn as a plane graph (with straight edges).

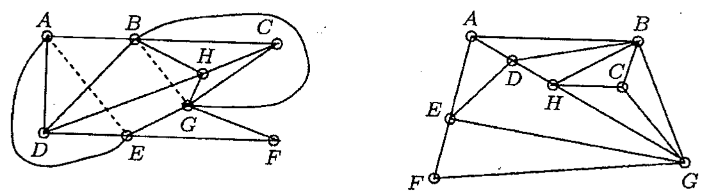


(b) This is not planar. There is a subgraph homeomorphic to $\mathcal{K}_5$ obtained by deleting the vertex of degree 3.

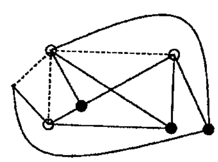
(c) This is planar.



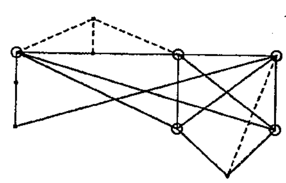
(d) This is planar.



(e) [BB] This is not planar. There is a subgraph homeomorphic to $\mathcal{K}_{3,3}$ as shown.



(f) This is not planar. It has a subgraph homeomorphic to $\mathcal{K}_5$ as shown.



10. (a) [BB] $\mathcal{K}_n$ is planar if and only if $n \leq 4$.

(b) $\mathcal{K}_{m,n}$ is planar if and only if $m \leq 2$ or $n \leq 2$.

13. Let $\mathcal{G}_1$ be obtained from $\mathcal{G}$ (with V vertices and E edges) by adding k vertices to edges. Then $V_1 = V+k$ and $E_1 = E+k$ (since there is one additional edge for each vertex added to an edge). Similarly, if $\mathcal{G}_2$ is obtained from $\mathcal{G}$ by adding ℓ vertices, then $V_2 = V + \ell$ and $E_2 = E + \ell$. The result now follows easily.

19. (a) [BB] Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n$ be the connected components of $\mathcal{G}$. Since $\mathcal{G}_i$ has at least three vertices, we have $E_i \leq 3V_i - 6$. Hence, $\sum E_i \leq 3 \sum V_i - 6n$, so $E \leq 3V - 6n$ as required.