HWK #2

12. (a)

(b) [BB] Try $p_n = c(8^n)$. We obtain $c(8^n) = 4[c(8^{n-1})] + 8^n$; that is, $8^n c = 4(8^{n-1})c + 8^n$, so $2^{3n}c = 2^{3n-1}c + 8^n$. Thus, $2^{3n-1}c = 8^n$, so $c = 2$ and $p_n = 2(8^n)$ is a particular solution. By (a), the homogeneous recurrence relation has solution $q_n = c(4^n)$, so $p_n + q_n = 2(8^n) + c(4^n)$. The initial condition gives $2 + c = 1$, so $c = -1$ and $a_n = 2(8^n) - 4^n$.

(c) [BB] For $n = 0$, $2(8^0) - 4^0 = 2 - 1 = 1$, as required.

Let $k \geq 0$ and assume that $a_k = 2(8^k) - 4^k$. We wish to prove that $a_{k+1} = 2(8^{k+1}) - 4^{k+1}$. Now

$$\begin{aligned} a_{k+1} &= 4a_k + 8^{k+1} = 4[2(8^k) - 4^k] + 8^{k+1} \quad \text{(by the induction hypothesis)} \\ &= 8^{k+1} - 4^{k+1} + 8^{k+1} = 2(8^{k+1}) - 4^{k+1} \end{aligned}$$

as desired. By the Principle of Mathematical Induction, the result is true for all $n \geq 0$.

15. (a) [BB] With $p_n = k$, we obtain $k = 4k - 9$, so $k = 3$. A particular solution is $p_n = 3$. The homogeneous recurrence $a_n = 4a_{n-1}$ has solution $q_n = c4^n$, so $a_n = p_n + q_n = c4^n + 3$ The initial condition gives $c + 3 = 4$ so $c = 1$ and we obtain $a_n = 4^n + 3$ as our solution.

(b) [BB] Try $p_n = (an + b)2^n$. We obtain

$$\begin{aligned} (an + b)2^n &= 4[a(n-1) + b]2^{n-1} + 3n2^n = 2(an + b - a)2^n + 3n2^n \\ an + b &= 2(an + b - a) + 3n \end{aligned}$$

Thus, $a = 2a + 3$ (so $a = -3$) and $b = 2b - 2a$ (so $b = 2a = -6$). Hence, a particular solution is $p_n = -(3n + 6)2^n$. The homogeneous recurrence $a_n = 4a_{n-1}$ has solution $q_n = 4^n$, so

$$p_n + q_n = c4^n - (3n + 6)2^n.$$

The initial condition gives $c - 6 = 4$ so $c = 10$ and we obtain $a_n = 10(4^n) - (3n + 6)2^n$ as our solution.

17. [BB] Try $p_n = a + bn + cn^2$. We obtain

$$\begin{aligned} a + bn + cn^2 &= 5[a + b(n-1) + c(n-1)^2] - 2[a + b(n-2) + c(n-2)^2] + 3n^2 \\ &= 5[a - b + c + (b - 2c)n + cn^2] - 2[a - 2b + 4c + (b - 4c)n + cn^2] + 3n^2 \\ &= 3a - b - 3c + (3b - 2c)n + (3c + 3)n^2. \end{aligned}$$

This gives the system

$$a = 3a - b - 3c; \quad b = 3b - 2c; \quad c = 3c + 3$$

with solution $a = -3$, $b = -\frac{3}{2}$, $c = -\frac{3}{2}$. Hence, a particular solution is $p_n = -3 - \frac{3}{2}n - \frac{3}{2}n^2$. The homogeneous recurrence $a_n = 5a_{n-1} - 2a_{n-2}$ has characteristic polynomial $x^2 - 5x + 2$ with characteristic roots $\frac{5 \pm \sqrt{17}}{2}$. Hence, $q_n = c_1(\frac{5-\sqrt{17}}{2})^n + c_2(\frac{5+\sqrt{17}}{2})^n$ and

$$p_n + q_n = -3 - \tfrac{3}{2}n - \tfrac{3}{2}n^2 + c_1\left(\tfrac{5-\sqrt{17}}{2}\right)^n + c_2\left(\tfrac{5+\sqrt{17}}{2}\right)^n.$$

The initial conditions give $-3 + c_1 + c_2 = 0$ and $-6 + c_1(\frac{5-\sqrt{17}}{2}) + c_2(\frac{5+\sqrt{17}}{2}) = 3$. This gives $c_1 = \frac{3}{2}(1 - \frac{1}{\sqrt{17}})$, $c_2 = \frac{3}{2}(1 + \frac{1}{\sqrt{17}})$ and, hence,

$$a_n = -3 - \tfrac{3}{2}n - \tfrac{3}{2}n^2 + \tfrac{3}{2}\left(1 - \tfrac{1}{\sqrt{17}}\right)\left(\tfrac{5-\sqrt{17}}{2}\right)^n + \tfrac{3}{2}\left(1 + \tfrac{1}{\sqrt{17}}\right)\left(\tfrac{5+\sqrt{17}}{2}\right)^n.$$