HWR 4

7. (a) [BB] Since $5n \le n^3$ for all $n \ge 3$, we can take $c = 1$, $n_0 = 3$, or $c = 5$, $n_0 = 1$.
   (b) For $n \ge 1$, we have $17n^4 + 8n^3 + 5n^2 + 6n + 1 \le 17n^4 + 8n^n + 5n^4 + 6n^4 + n^4 = 37n^4$. So we can take $c = 37$ and $n_0 = 1$.
   (c) [BB] For $n \ge 1$, we have $8n^3 + 4n^2 + 5n + 1 \le 9n^4 + 18n^2 + 24n + 6 = 3(3n^4 + 6n^2 + 8n + 2)$, so we can take $c = 3$, $n_0 = 1$.
   (d) For $n \ge 1$, $2^n \le 3^n$, so take $c = 1$, $n_0 = 1$.

14. (a) We proceed by induction on n. Since $10^3 = 1000$ and $2^{10} = 1024$ the statement holds for $n = 10$. Now suppose the statement is true for some $k > 10$. Then

$$\begin{aligned}
(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\
&\le k^3 + 3k^2 + 3k^2 + k^2 \\
&\le k^3 + 7k^2 \\
&< k^3 + k^3 = 2k^3 && \text{since } 7 < k \\
&\le 2(2^k) = 2^{k+1} && \text{by the induction hypothesis}
\end{aligned}$$

which is the statement for $n = k + 1$. By the Principle of Mathematical Induction, the statement is true for all $n \ge 10$.

   (b) Since $n^2 < n^3$ and $n^3 < 2^n$ for $n \ge 10$, certainly $n^2 < 2^n$ for $n \ge 10$. (In fact, as mentioned in this section, $n^2 < 2^n$ for $n \ge 5$.) Thus, $n^2 = \mathcal{O}(2^n)$. We have now to prove that $2^n \ne \mathcal{O}(n^2)$. Assume to the contrary that $2^n = \mathcal{O}(n^2)$. Then $2^n < cn^2$ for some constant c and all sufficiently large enough n. Since $2^n > n^3$ for sufficiently large n (by part (a)), we would have $n^3 < 2^n < cn^2$ and hence $n < c$ for large n. This is clearly not the case.

19. (a) By Proposition 8.2.7, f has the same order as n^4.
   (b) [BB] Since $\log n = \mathcal{O}(n)$, $n \log n = \mathcal{O}(n^2)$, hence, $f = \mathcal{O}(n^2)$. Since $n^2 \le f(n)$ for all $n \ge 1$, $f \asymp n^2$.
   (c) $5 \asymp 1$ is a direct consequence of Exercise 15 with $f(n) = 1$, $k = 5$.
   (d) Since $2^n = \mathcal{O}(n^n)$ and $n^n \asymp 4n^n$, we have that $2^n = \mathcal{O}(4n^n)$ by transitivity of $\mathcal{O}$ (Exercise 10(a)) and so $f(n) = \mathcal{O}(n^n)$ by Proposition 8.2.3. Since $n^n \le f(n)$ for any $n \ge 1$, we have easily $n^n = \mathcal{O}(f(n))$ and so $f(n) \asymp n^n$.