How to solve a homogeneous recurrence relation

Consider the following second order (also known as second degree) homogeneous linear recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0 \tag{2}$$

with constant coefficients.

Step 1. Solve the characteristic equation $x^2 + c_1 x + c_2 = 0$. Let x_1, x_2 be the roots.

Step 2. If $x_1 \neq x_2$, then the general solution is of the form $a_n = K_1 x_1^n + K_2 x_2^n$. If $x_1 = x_2$, then the general solution is of the form $a_n = K_1 x_1^n + K_2 n x_1^n$.

Step 3. If initial conditions are given, then solve for the constants K_1, K_2 using the initial conditions.

How to find a particular solution

A particular solution of

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} = P(n)\beta^n, \tag{3}$$

where $P(n)$ is a polynomial and β is a constant can be found of the form

$$a_n = Q(n)\beta^n,$$

where $Q(n)$ is a polynomial of degree $k + m$, where k is the degree of P and m is the multiplicity of β as a root of the characteristic equation of (3); that is, as a root of $x^2 + c_1 x + c_2 = 0$.

Example. $a_n - a_{n-1} = n$
Here $x - 1 = 0$ is the characteristic equation, $P(n) = n$ and $\beta = 1$. Thus a solution can be found of the form $a_n = An^2 + Bn + C$.

Example. $a_n - a_{n-1} = n2^n$
Again, $x - 1 = 0$ is the characteristic equation, $P(n) = n$ and $\beta = 2$. Thus a solution can be found of the form $a_n = (An + B)2^n$.

Example. $a_n - 2a_{n-1} + a_{n-2} = n^2 + n + 2$
The characteristic equation is $x^2 - 2x + 1 = 0$, $P(n) = n^2 + n + 2$ and $\beta = 1$. The multiplicity is 2, and so a solution can be found of the form $a_n = An^4 + Bn^3 + Cn^2 + Dn + E$.

How to solve a non-homogeneous recurrence relation

To find all solutions of equation (3) we proceed as follows:

Step 1. Solve the corresponding homogeneous recurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} = 0$. Let a_n^H denote the solution. It will involve constants K_1, K_2.

Step 2. Find one particular solution of the equation (3) as described above, and denote it by a_n^P.

Step 3. The general solution of (3) is $a_n = a_n^H + a_n^P$.

Step 4. If initial conditions are given, solve for K_1, K_2 using initial conditions.