NAME: Key

Math 2602 Test 1 September 16, 2009 TA:

Instructions. Please write your name legibly on top of each page. Closed book, closed notes. Small nonprogrammable calculators are allowed. To receive credit you must show your work and justify your answers. Please circle or box your answer. Good luck!

1. (15 points) Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 9$.

The characteristic equation is $x^2 - 6x + 9 = 0$. It has a root $x_1 = x_2 = 3$ of multiplicity two. The general solution is of the form

$$a_n = K_1 3^n + K_2 n 3^n$$

We need to solve for K_1 and K_2 from the initial conditions. We have

$$1 = a_0 = K_1$$
$$9 = a_1 = 3K_1 + 3K_2$$

So $K_1 = 1$ and $K_2 = 2$. The solution is

$$a_n = 3^n + 2n3^n = (2n+1)3^n$$

2. (15 points) Solve the recurrence relation $a_n = a_{n-1} + 4n + 1$, $a_0 = 4$.

The homogeneous recurrence relation is $a_n - a_{n-1} = 0$. The characteristic equation is $x - 1 = 0$ with root $x = 1$ of multiplicity one. Thus a solution to the homogeneous recurrence relation is $a_n^H = K$, where K is a constant. We will look for a particular solution of the form $a_n^P = An^2 + Bn + C$ (degree of $4n+1$ plus the multiplicity of 1 as a root of char. eq.)

$$An^2 + Bn + C = A(n-1)^2 + B(n-1) + C + 4n + 1$$
$$\cancel{An^2} + \cancel{Bn} + \cancel{C} = \cancel{An^2} - 2An + A + \cancel{Bn} - B + \cancel{C} + 4n + 1$$
$$(4-2A)n + A - B + 1 = 0$$

So $4 - 2A = 0$ and $A - B + 1 = 0$. We get $A = 2$, $B = 3$ and can set $C = 0$.

Thus $a_n^P = 2n^2 + 3n$. Now

$a_n = a_n^H + a_n^P = K + 2n^2 + 3n$. From the initial condition $4 = a_0 = K$, and so

$$a_n = 2n^2 + 3n + 4$$

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4. (30 points) Find a function in the list

$$5, \log n, n\log n, n^2 \log n, n, n^2, n^3, n(\log n)^2, 2\ \ , 2\ \ \log n, n!, n$$

that has the same order as the given function. **Justify your answer**.

(a) $4n^3 + 7n^2 - 3n + 6\log n$

n^3

$$\lim_{n\to\infty} \frac{4n^3+7n^2-3n+6\log n}{n^3} = \lim_{n\to\infty}\left(4+\frac{7}{n}-\frac{3}{n^2}+\frac{6\log n}{n^3}\right)=4$$

(b) $3n^2 \log n + 5n(\log n)^2$

$n^2 \log n$

$$\lim_{n\to\infty} \frac{3n^2\log n+5n(\log n)^2}{n^2\log n} = \lim_{n\to\infty}\left(3+\frac{5\log n}{n}\right)=3$$

(c) $2\ \ + n^2 + \log n$

2^n

$$\lim_{n\to\infty} \frac{2^n+n^2+\log n}{2^n} = \lim_{n\to\infty}\left(1+\frac{n^2}{2^n}+\frac{\log n}{2^n}\right)=1$$

5. (20 points) Find a formula for $1 + 4 + 7 + \cdots + 3n - 2$.

Let $a_n = 1 + 4 + \cdots + 3n - 2$. Then

$$a_n = a_{n-1} + 3n - 2 \quad \text{and} \quad a_1 = 1$$

$a_n^H = K$ characteristic equation $x - 1 = 0$ has root $x = 1$

$$a_n^P = An^2 + Bn + C$$

$$An^2 + Bn + C = A(n-1)^2 + B(n-1) + C + 3n - 2$$

$$\cancel{An^2} + \cancel{Bn} + \cancel{C} = \cancel{An^2} - 2An + A + \cancel{Bn} - B + \cancel{C} + 3n - 2$$

$$(3 - 2A)n + A - B - 2 = 0$$

$A = \frac{3}{2}$, $B = -\frac{1}{2}$, may let $C = 0$

$$a_n^P = \frac{3}{2}n^2 - \frac{1}{2}n$$

$$a_n = a_n^H + a_n^P$$

$$a_n = K + \frac{3}{2}n^2 - \frac{1}{2}n$$

From initial condition

$$1 = a_1 = K + \frac{3}{2} - \frac{1}{2}$$

$$K = 0$$

$$a_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{n(3n-1)}{2}$$

6. (20 points) Prove that $\frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^n} = 1 - \frac{1}{3^n}$ holds for all $n \geq 1$.

Basis step For $n = 1$ we have $\frac{2}{3} = 1 - \frac{1}{3}$

Induction step Let $k \geq 1$. We assume

$$\frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^k} = 1 - \frac{1}{3^k} \quad \text{(induction hypothesis)}$$

We must show

$$\frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^{k+1}} = 1 - \frac{1}{3^{k+1}}$$

We have

$$\underbrace{\frac{2}{3} + \frac{2}{9} + \cdots + \frac{2}{3^k}}_{1 - \frac{1}{3^k}} + \frac{2}{3^{k+1}} \underset{\text{by the induction hypothesis}}{=} 1 - \frac{1}{3^k} + \frac{2}{3^{k+1}} = 1 - \frac{3}{3^{k+1}} + \frac{2}{3^{k+1}} = 1 - \frac{1}{3^{k+1}},$$

as desired.