Application to Algebra

A ***ring*** is $(R, +, \cdot, 0, 1)$, where

- $(R, +)$ is an abelian group with identity 0
- $a(bc) = (ab)c$ and 1 is a multiplicative identity
- $(a + b)c = ac + bc$ and $c(a + b) = ca + cb$

Examples. (i) Z

(ii) Matrices over any ring

The ***commutator*** in a ring is $[a, b] = ab - ba$. More generally,

$$[a_1, a_2, \ldots, a_k] = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)}$$

where the summation is over all permutations σ of $\{1,2, \ldots, k\}$.

If $[a_1, a_2, \ldots, a_k] = 0$ for all $a_i \in R$, then R is said to satisfy the ***k^{th} polynomial identity***.

THEOREM (Amitsur, Levitzky) The ring of $k \times k$ matrices over a commutative ring R satisfies the $(2k)^{\text{th}}$ polynomial identity. In other words, if $A_1, A_2, \ldots, A_{2k}$ are $k \times k$ matrices with entries in R, then $[A_1, A_2, \ldots, A_{2k}] = 0$.

Proof. Since $[A_1, A_2, \dots, A_{2k}]$ is linear in each variable, it suffices to prove the theorem for matrices of the form E_{ij} (each entry 0, except $e_{ij} = 1$). So we must show

$$[E_{i_1 j_1}, E_{i_2 j_2}, \dots, E_{i_{2k} j_{2k}}] = 0.$$

Define a directed multigraph D by $V(D) = \{1,2, \dots, k\}$ and $E(D) = \{e_1, e_2, \dots, e_{2k}\}$, where $e_t = i_t j_t$. Then

$$E_{\sigma(e_1)} E_{\sigma(e_2)} \cdots E_{\sigma(e_{2k})} \neq 0$$

if and only if $\sigma(e_1), \sigma(e_2), \dots, \sigma(e_{2k})$ is the edge-set of an Euler trail in D. So

$$\left[E_{i_1 j_1}, E_{i_2 j_2}, \dots, E_{i_{2k} j_{2k}}\right] = \left[E_{e_1}, E_{e_2}, \dots, E_{e_{2k}}\right] =$$

$$\sum_{\sigma} \operatorname{sgn}(\sigma) E_{\sigma(e_1)} E_{\sigma(e_2)} \cdots E_{\sigma(e_{2k})} = \sum_{W} \operatorname{sgn}(W) E_{xy}$$

where are the last summation is over all Euler trails, $\operatorname{sgn}(W)$ is the sign of the permutation of $E(D)$ determined by W, x is the origin of W and y is its terminus. Furthermore,

$$\sum_{W} \operatorname{sgn}(W) E_{xy} = \sum_{x,y \in V(D)} \left(\sum_{W} \operatorname{sgn}(W) \right) E_{xy}$$

where the last summation is over all Euler trails from x to y.

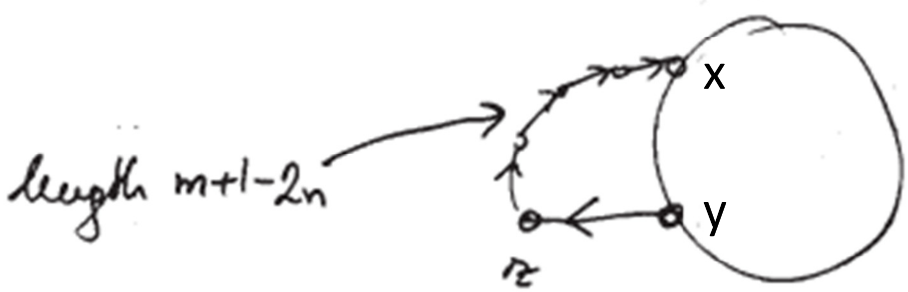
Lemma. Let D be a directed multigraph with $|E(D)| \geq 2|V(D)|$ and the let $x, y \in V(D)$. Then

$$\varepsilon(D, x, y) \coloneqq \sum_W \operatorname{sgn}(W) = 0$$

where the summation is over all Euler trails from x to y.

Proof. Let $n = |V(D)|$ and $m = |E(D)|$. WMA no isolated vertices.

Step 1. We show that it suffices to prove the theorem for $x = y$ and $m = 2n$. We construct a directed multigraph D' by adding a new vertex z, an edge yz and a path $z \to x$ of length $m + 1 - 2n$.



Then $|V(D')| = n + m + 1 - 2n = m + 1 - n$

$$|E(D')| = m + m + 1 - 2n + 1 = 2(m + 1 - n)$$

$$|\varepsilon(D, x, y)| = |\varepsilon(D', z, z)|$$

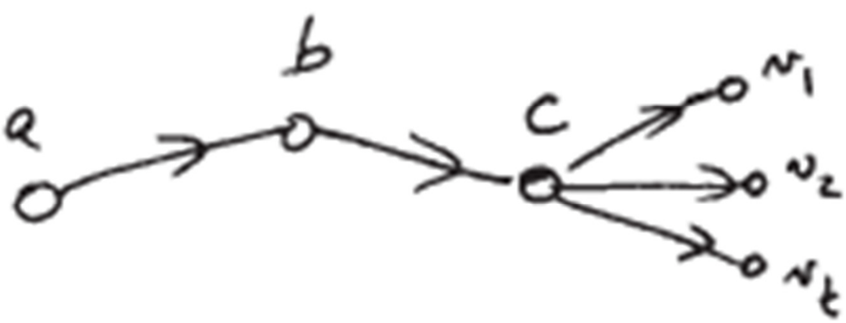
We proceed by induction on n. WMA D has a closed Euler trail, for otherwise $\varepsilon(D,z,z)=0$. Thus $\deg^+(v)=\deg^-(v)$ for all $v\in V(D)$.

Step 2. If D has a parallel edge, then $\varepsilon(D,z,z)=0$.

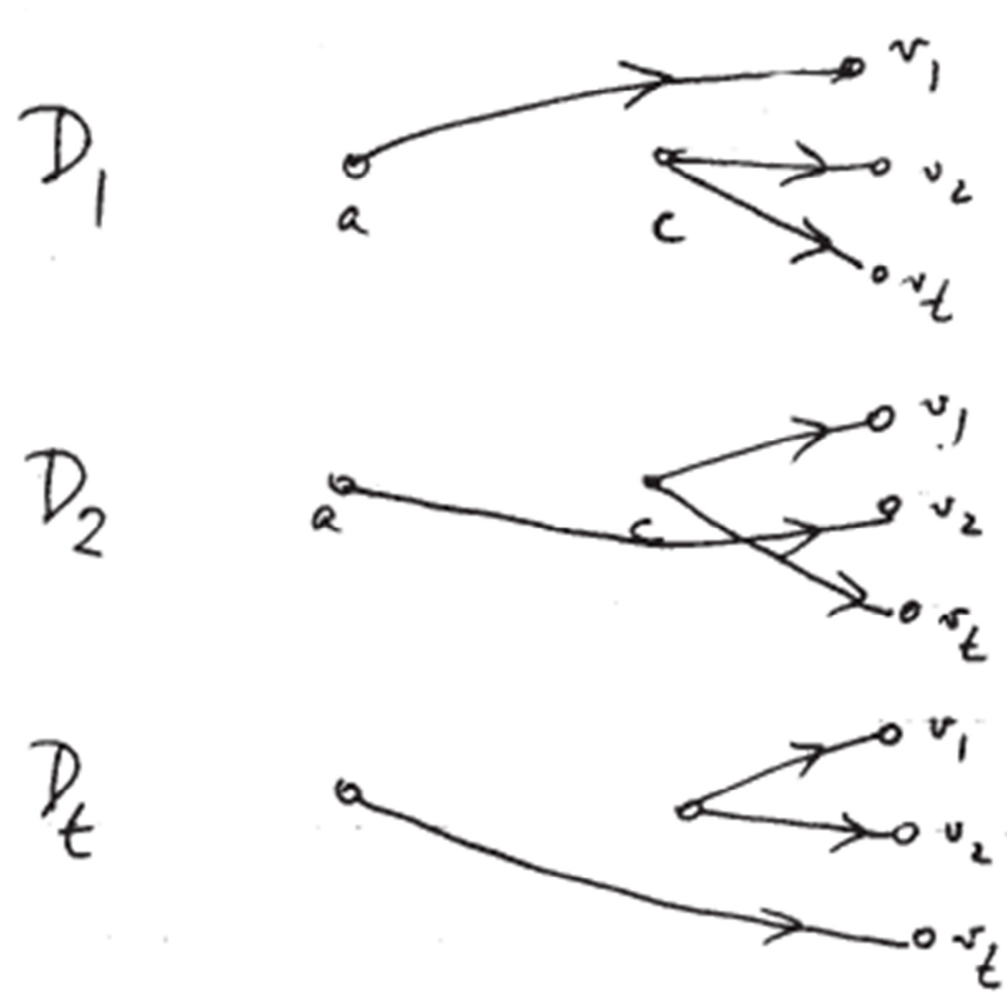
Step 3. If D has a vertex $b\neq z$ of degree 2 with one neighbor

Delete b and go by induction

Step 4. If D has a vertex $b\neq z$ of degree 2 with two neighbors



Define



Then

$$|\varepsilon(D,z,z)| = \left|\sum_{i=1}^{t} \varepsilon(D_i,z,z)\right|$$

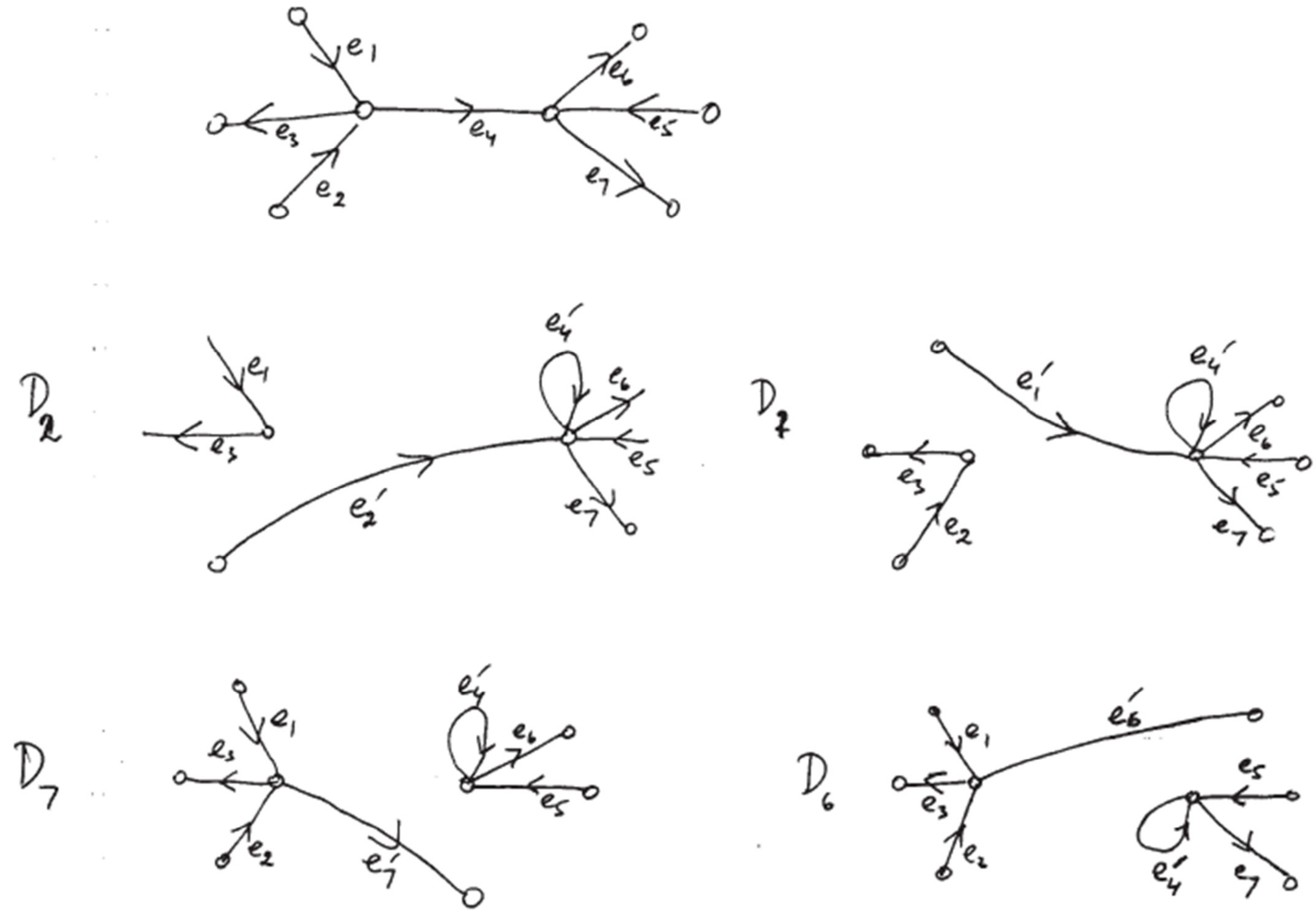
Step 5. If D has a loop at a vertex $b \neq z$ of degree 4

$$|\varepsilon(D, z, z)| = |\varepsilon(D', z, z)|$$

Step 6. If none of the above apply, then either

- $\deg^+(v) = \deg^-(v) = 2$ for all $v \in V(D)$, or
- $\deg^+(z) = 1,\ \deg^+(w) = 3, \deg^+(v) = 2$ for other v.

Step 7. There exist two adjacent vertices of outdegree two (exercise)



$$\varepsilon(D,z,z) = \varepsilon(D_1,z,z) + \varepsilon(D_2,z,z) - \varepsilon(D_6,z,z) - \varepsilon(D_7,z,z)$$

Question. Is the bound $|E(D)| \geq 2|V(D)|$ in the lemma best possible?