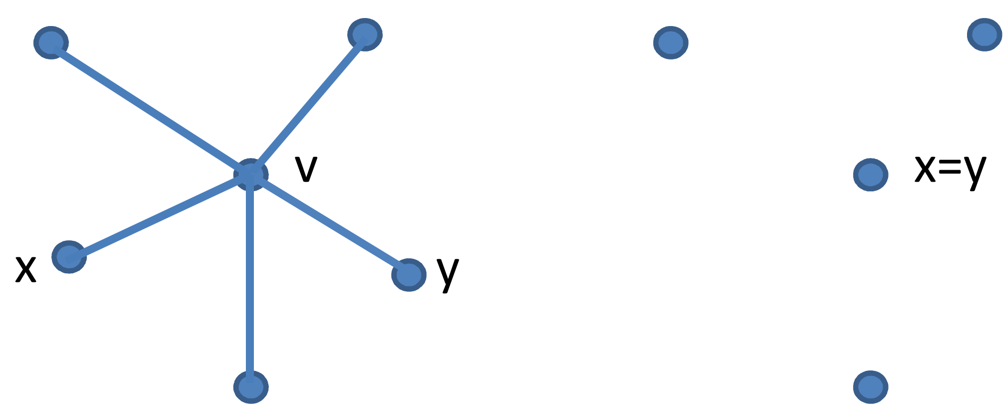
Corollary. Every simple planar graph G has a vertex of degree at most five.

Proof. We may assume G has $n \geq 3$ vertices. Then the sum of the degrees is $2|E(G)| \leq 6n - 12$ by Corollary 1.14, and hence G has a vertex of degree at most five.

Corollary. Every planar graph is 5-colorable.

Proof. Let v be a vertex of G of degree at most five. If v has degree at most four, then the result follows by induction applied to $G\backslash v$. So WMA v has degree five. Since G has no K_5 subgraph by planarity, some two neighbors of v, say x and y, are not adjacent. Apply induction to the graph obtained from $G\backslash v$ by identifying x and y.

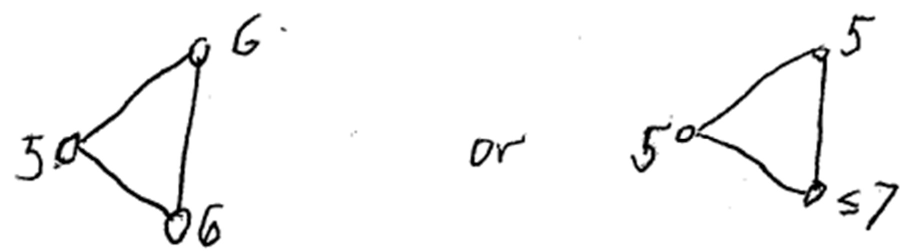


Theorem. Every planar graph is 4-colorable.

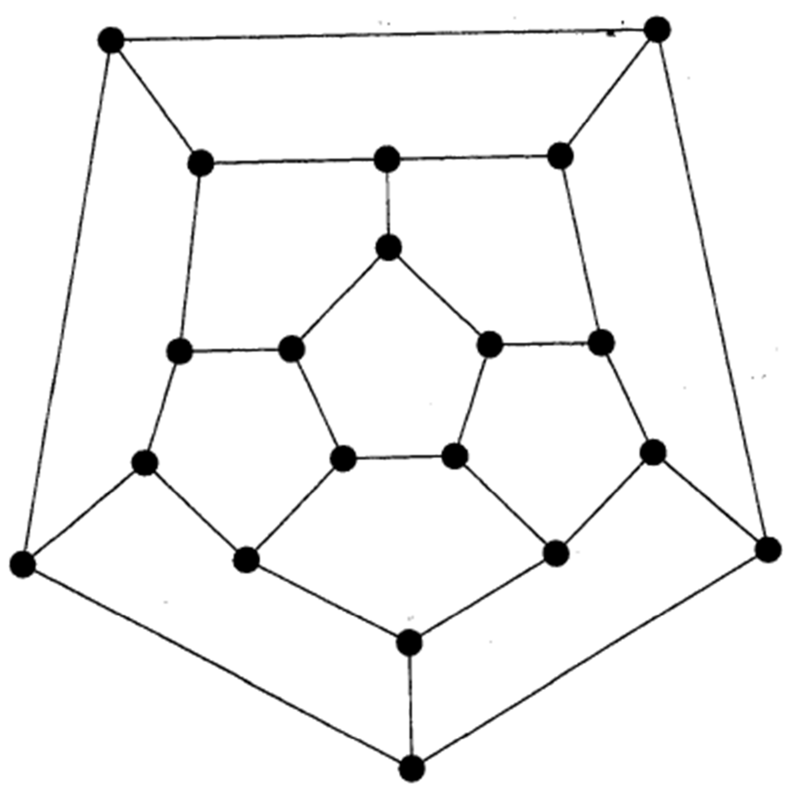
THE DISCHARGING METHOD

THEOREM. Every simple plane graph of minimum degree five has a face bounded by a triangle $C = xyz$ such that

$$\deg(x) + \deg(y) + \deg(z) \leq 17.$$

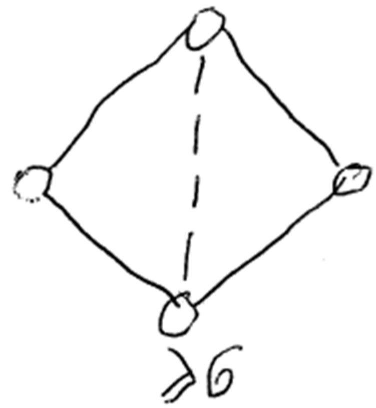


EXAMPLE. 17 is best possible



Dodecahedron

PROOF. We will allow G to be a multigraph, but insist every face have length ≥3. Suppose for a contradiction that said face does not exist. We may assume that $E(G)$ is maximal. It follows that every vertex v incident with a ≥4-face has degree five, for otherwise we can add an edge.



Assign charges as follows:

$$\mathrm{ch}(v) := \deg(v) - 6$$

$$\mathrm{ch}(f) := 2|f| - 6$$

Then the sum of all charges is

$$\sum \mathrm{ch}(v) + \sum \mathrm{ch}(f) = \sum \deg(v) - 6|V(G)| + 2\sum|f| - 6|F(G)| =$$

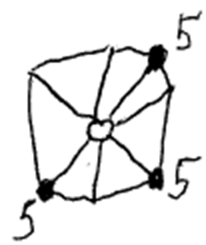
$$2|E(G)| - 6|V(G)| + 4|E(G)| - 6|F(G)| =$$

$$= 6(|E(G)| - |V(G)| - |F(G)|) = -12$$

We now redistribute the charges as follows:

(1) Every ≥4-face sends 1/2 to every incident vertex (of degree five)

(2) Every 7-vertex sends 1/3 to every neighbor of degree five

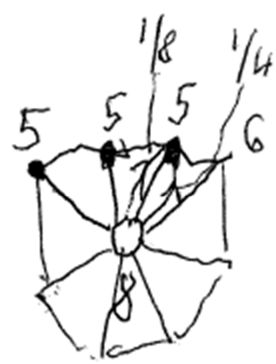


(3) Every ≥8 vertex v sends 1/4 or 0 through every incident triangular face vxy:

if deg(x)=deg(y)=5, then both x,y receive 1/8

if one of x,y has degree five, then it gets 1/4

if x,y have degree ≥6, then no charge is sent



Let ch'(v) and ch'(f) denote the new charges. Then

$$\sum \mathrm{ch}'(v) + \sum \mathrm{ch}'(f) = \sum \mathrm{ch}(v) + \sum \mathrm{ch}(f) = -12$$

But clearly $\mathrm{ch}'(f) \geq 0$ for every face f.

We claim that $\mathrm{ch}'(v) \geq 0$ for every vertex v, which gives a contradiction.

$\mathrm{ch}'(v) = 0$ for every v of degree 6

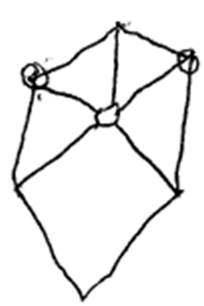
$\mathrm{ch}'(v) \geq 0$ for every v of degree 7, because every incident face is a triangle and at most 3 neighbors have degree 5.

$\mathrm{ch}'(v) \geq 0$ for every v of degree ≥8, because it sends ≤1/4 through every incident face

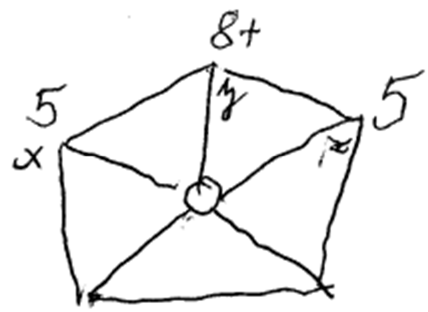
$\mathrm{ch}'(v) \geq 0$ for every v of degree 5:

If at least two incident ≥4-faces, then receive 1/2 from each

If one incident ≥4-face, then receive 1/2 from it and at least two neighbors have degree ≥7, each sending ≥1/4



If every incident face is a triangle, then at least three neighbors have degree $\geq$7. WMA one neighbor, say y, sends only 1/4. Then x and z have degree 5, and the remaining two neighbors send $\geq$3/8 each.



This proves the theorem.

Other uses:

$$\mathrm{ch}(v) := 2\deg(v) - 6$$

$$\mathrm{ch}(f) := |f| - 6$$

Then the sum of the charges is -12

or

$$\mathrm{ch}(v) := \deg(v) - 4$$

$$\mathrm{ch}(f) := |f| - 4$$

Then the sum of the charges is -8.

Corollary. There is no planar graph of minimum degree at least 4 and every face of size at least 4.