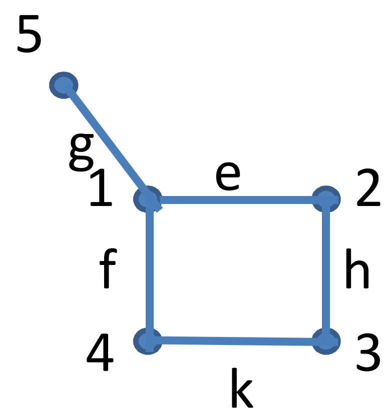
A ***graph*** is a pair (V, E), where V is a (finite) set and E is a set of 2-element subsets of V.

Example. $V = \{1,2,3,4,5\}$

$E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{1,5\}\}$

Then (V, E) is a graph.



V = set of vertices

E = set of edges

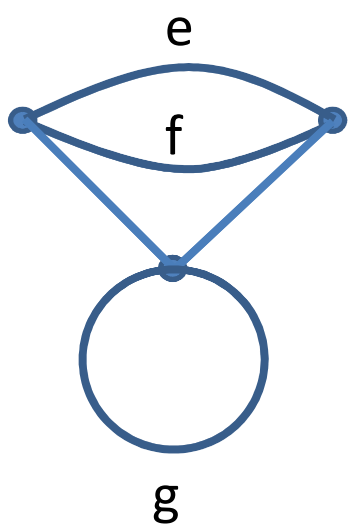
Two vertices $u, v \in V$ are ***adjacent*** if $\{u, v\} \in E$. We will denote that by $u \sim v$.

The vertex u is ***incident*** with the edge $e = \{u, v\}$.

If $e = \{u, v\}$, then u, v are the ***ends*** (sometimes called ***endpoints***) of e.

Two edges e, f are ***adjacent*** if they have a common end.

Example. $1 \sim 2$, $1 \nsim 3$, 1 is incident with e, not with h.



A ***multigraph*** is a triple (V, E, ψ), where V, E are finite sets and ψ is an incidence relation that assigns to every edge $e \in E$ a pair of not necessarily distinct vertices of V, called its ***ends***.

Two edges are ***parallel*** if they have the same ends.

An edge is a ***loop*** if its ends are equal.

A graph can be regarded as a multigraph. A multigraph with no loops or parallel edges it can be regarded as a graph.

Notation. If G is a graph or multigraph, then
$V(G) =$ set of vertices of G
$E(G) =$ set of edges of G

Let G be a graph or multigraph. The ***degree*** of a vertex $v \in V(G)$, denoted by $\deg_G(v)$ or $\deg(v)$, is the number of edges of G incident with v, counting loops twice.

Lemma. In any multigraph G

(i) $\sum_{v \in V(G)} \deg_G(v) = 2|\mathrm{E}(\mathrm{G})|$

(ii) the number of vertices of odd degree is even.

Proof. Easy

A ***walk*** in a multigraph G is a sequence

$$W = (v_0, e_1 v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k)$$

where e_i has ends v_{i-1}, v_i for all $i = 1,2, \dots, k$. The ***length*** of W is k (the number of edges). The vertices v_0, v_k are the ***ends***, and W is a v_0-v_k-walk. It is a ***closed walk*** if $v_0 = v_k$. It is a ***trail*** if it has no repeated edges.

A ***path*** is a graph (V, E), where $V = \{v_0, v_1, \dots, v_k\}$ and $E = \{\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{k-1}, v_k\}\}$. This is essentially a walk with no repeated vertices or edges.

A ***cycle*** is a graph (V, E), where $V = \{v_1, v_2, \dots, v_k\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{k-1}, v_k\}, \{v_k, v_1\}\}$. This is essentially a closed walk with no repeated vertices or edges, except the first and last vertex.

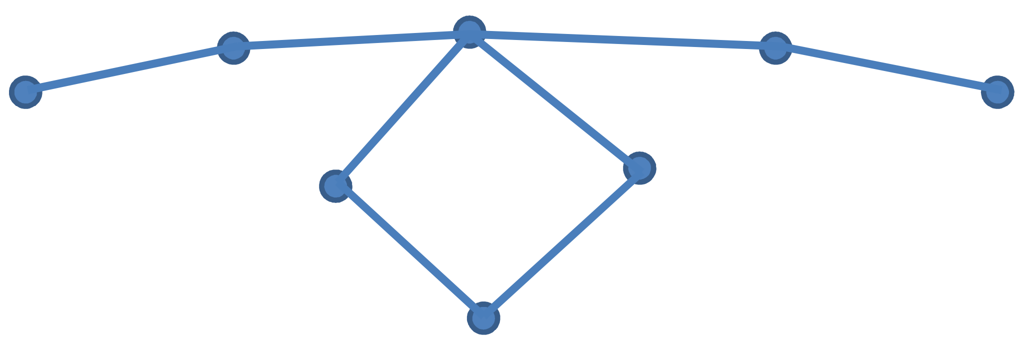
Lemma. Let G be a multigraph.

(i) Every u-v walk includes the edge-set of a u-v path.

(ii) Every closed trail includes the edge-set of a cycle.

(iii) Every closed walk of odd length includes the edge-set of an odd cycle.

Proof. (i) Let W be a u-v walk. Let W' be a subsequence of W that is itself a u-v walk and is as short as possible.



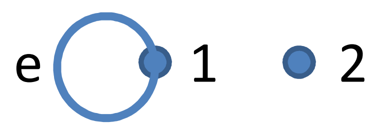
(ii) exercise, (iii) exercise

A graph H is a ***subgraph*** of a graph G if
$V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

When is H a submultigraph of a multigraph G? Is it enough that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$?

Example.

H: 

G: 

A multigraph is H a ***submultigraph*** of a multigraph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and the incidences in H are the same as in G.

A multigraph G is ***connected*** if for all $u, v \in V(G)$ there is a u-v path in G.

A (connected) ***component*** of G is a maximal connected subgraph.

A multigraph G is ***bipartite*** if $V(G)$ can be partitioned into disjoint sets A, B such that every edge has one end in A and the other in B.

Examples:

Theorem. A multigraph is bipartite if and only if it has no odd cycle.

Proof. $\Rightarrow$ straightforward

$\Leftarrow$ Let G be a multigraph with no odd cycle. WMA G is connected, for if each component is bipartite, then G is. Let $v \in V(G)$ and let

$$A = \{u \in V(G) \colon \exists \text{ even } u - v \text{ walk in } G\}$$

$$B = \{u \in V(G) \colon \exists \text{ odd } u - v \text{ walk in } G\}$$

Since G is connected, $A \cup B = V(G)$. We have $A \cap B = \emptyset$, for o.w. G has a closed odd walk and hence an odd cycle by a previous lemma, a contradiction. Clearly each edge has one end in A and the other in B. $\square$