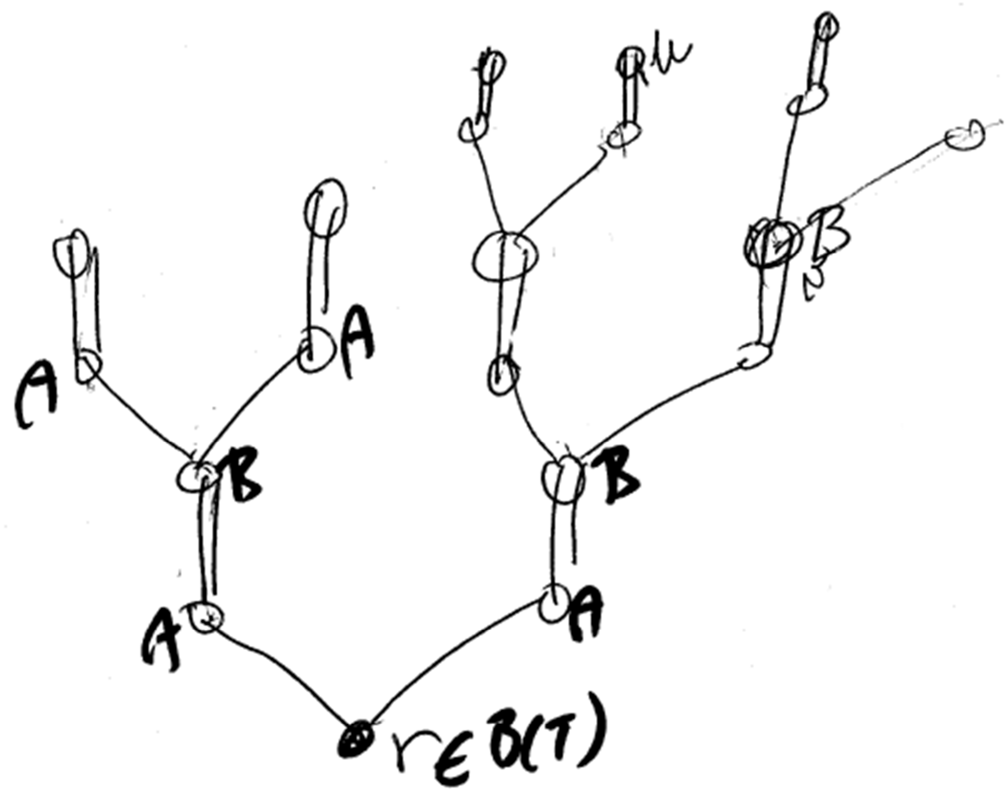
Definition. Let M be a matching in G, and let $r \in V(G)$ be M-unsaturated. An M-**alternating tree rooted at** r is a tree T such that

(i) T is a subgraph of G

(ii) $r \in V(T)$

(iii) every path in T with end r is M-alternating

(iv) if $e \in M$ is incident with a vertex of T, then $e \in E(T)$



Given an M-alternating tree T let

$$A(T) := \{v \in V(T) : v \text{ is at odd distance from } r \text{ in } T\}$$

$$B(T) := \{v \in V(T) : v \text{ is at even distance from } r \text{ in } T\}$$

Theorem. Let G be a graph, let M be a maximum matching in G, let $r \in V(G)$ be M-unsaturated, and let T be an M-alternating tree rooted at r. Then there exists a set $X \subseteq V(G)$ such that $X \cap V(T) \subseteq A(T)$ and $o(G \backslash X) > |X|$.

Proof. WMA that T is a maximal M-alternating tree.

Claim 1. Every $u \in A(T)$ is incident with an edge of M, which belongs to T by (iv).

Proof. Otherwise the path in T from u to r is M-augmenting, contrary to the maximality of M.

Claim 2. If $e = uv \in E(G)$ and $u \in B(T)$, then $v \in V(T)$.

Proof. Suppose for a contradiction that $v \notin V(T)$. The first edge on the path from u to r belongs to M, and so $e \notin M$. Let T' be obtained from T by adding the edge e. If v is not M-saturated, then T' is M-alternating, contrary to the maximality of T. Otherwise, let T'' can be obtained from T' by adding the edge of M incident with v; then T'' contradicts the maximality of T. This proves the claim.

It follows from the claims that if G has no $B(T)$- $B(T)$ edge, then $X := A(T)$ satisfies the conclusion of the theorem, because every vertex in $B(T)$ is isolated in $G \backslash X$.

So WMA $\exists\, uv \in E(G)$ with $u, v \in B(T)$. The edge uv creates a unique M-blossom C. Let $G' := G/C$, $M' := M - E(C)$, $T' := T/C$. Then M' is a maximum matching in G' by the previous lemma, and T' is an M'-alternating tree in G'.

By induction applied to G', M', T' there is a set $X \subseteq V(G')$ such that $X \cap V(T') \subseteq A(T')$ and $o(G' \backslash X) > |X|$. We have

$$X \cap V(T) \subseteq X \cap V(T') \subseteq A(T') \subseteq A(T)$$

Let $k := |X| + 1$ and let $C_1', \ldots, C_k'$ be distinct odd components of $G' \backslash X$. Let w be the new vertex of G'. Then $w \in B(T')$. Note that $w \notin X$, because $X \cap V(T') \subseteq A(T')$. Let

$$C_i := \begin{cases} C_i' & \text{if } w \notin V(C_i') \\ G[V(C_i') - \{w\} \cup V(C)] & \text{otherwise} \end{cases}$$

Then $C_1, \ldots, C_k$ are odd components of $G \backslash X$, and hence $o(G \backslash X) > |X|$, as desired. $\square$

An algorithm

Input: A graph G and a matching M

Output: Either a matching of size $|M| + 1$, or a set $X \subseteq V(G)$ such that $o(G \backslash X) > |X|$.

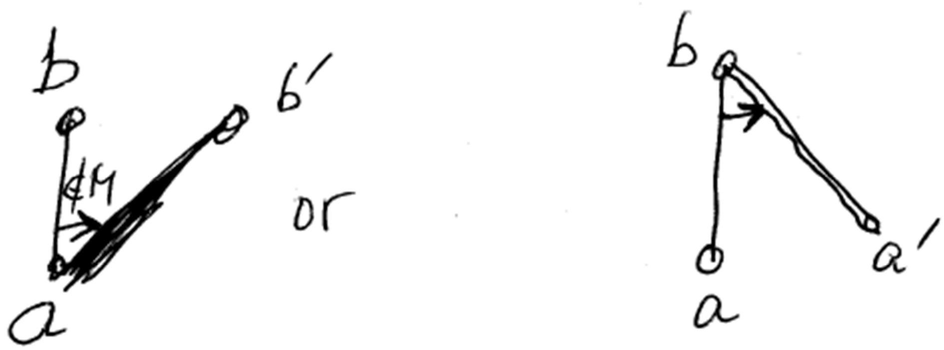
Description. Construct a maximal M-alternating tree T. If some $u \in A(T)$ is not incident with an edge of M, then we construct a larger matching. So WMA Claim 1 holds. Claim 2 holds, because it does not use the maximality of M. If there is no $B(T)$- $B(T)$ edge, then we output the set $X := A(T)$ and stop. Otherwise we find a blossom, construct G' and M' as in the proof and apply the algorithm recursively to G' and M'. If the recursive call returns a set X, then we return it and stop. If the recursive call returns a matching larger than M', then we use the proof of the previous lemma to convert that matching to a matching in G of size $|M| + 1$.

By iterating the above algorithm we obtain an algorithm to decide whether there is a perfect matching in an input graph G, and to construct one if one exists.

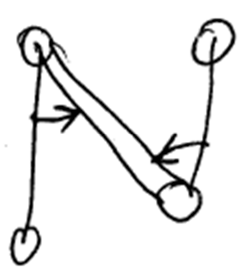
Stable matchings

Let G be a bipartite graph with bipartition (A, B) and let us assume that for every $v \in V(G)$ there is a specified linear ordering of edges incident with v ("preferences"). A matching M is **stable** if for every edge $ab \in E(G) - M$ either

- $\exists b' \in B$ such that $ab' \in M$ and a prefers b' to b, or
- $\exists a' \in A$ such that $a'b \in M$ and b prefers a' to a.



Example.



Theorem (Gale, Shapley 1960s). For every bipartite graph and every assignment of preferences there is a stable matching.

Proof. In an odd round each boy proposes to his highest preference among girls who have not yet rejected him. In an even round each girl refuses all who proposed but her highest suitor. This stops when no girl refuses a suitor; then each girl marries her unique suitor if she has one.

Pick $ab \in E(G) - M$. We must show that ab satisfies one of the requirements from the definition of a stable matching.

WMA that either a is not married or is married to b', who he likes less than b. That means a at some point proposed to b and got rejected. At that point b had a better suitor, and therefore ended up with a suitor she likes better than a. □