A **partially ordered set** (**poset**) is a pair $(X, \leq)$, where X is a (finite) set and $\leq$ is a partial order; that is $\leq$ is

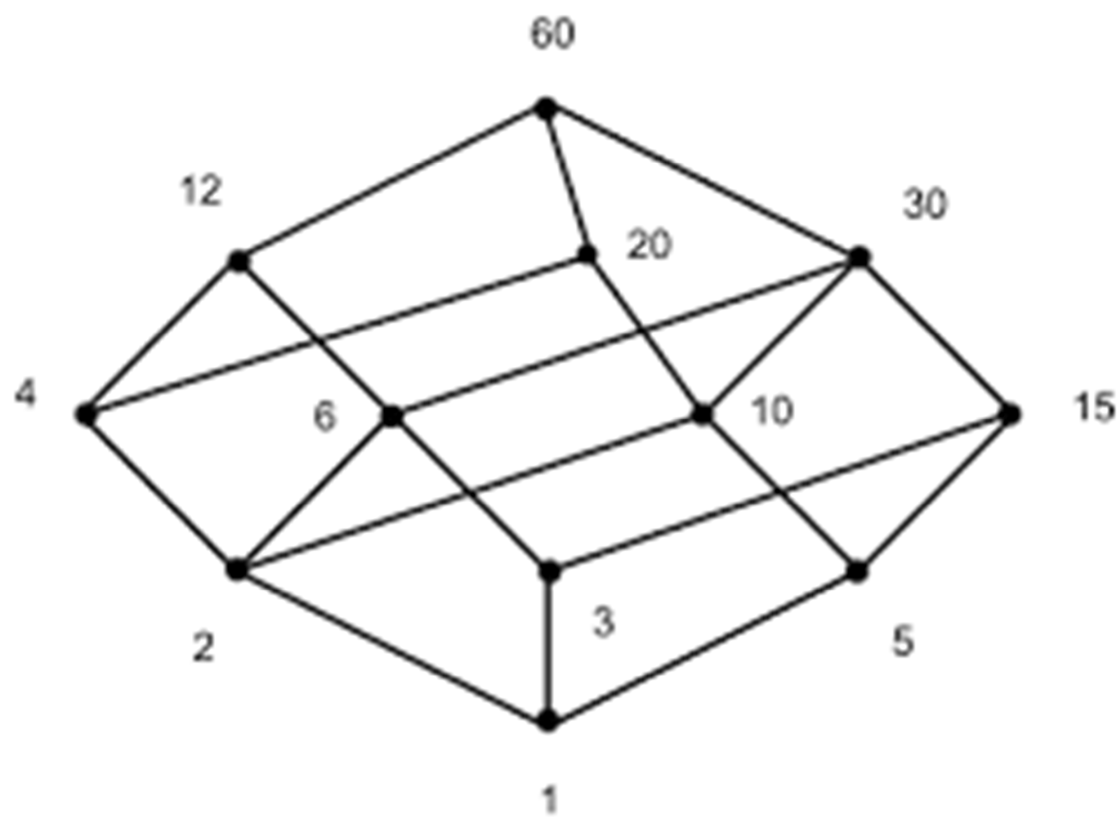
- reflexive: $x \leq x \quad \forall\, x \in X$

- transitive: $x \leq y$ and $y \leq z \Rightarrow x \leq z \quad \forall\, x, y, z \in X$

- antisymmetric: $x \leq y$ and $y \leq x \Rightarrow x = y \quad \forall\, x, y \in X$.

A **chain** is a set $C \subseteq X$ such that $x \leq y$ or $y \leq x\ \forall\, y, x \in C$. An **antichain** is a set $A \subseteq X$ such that $x \nleq y$ for all distinct $x, y \in X$.

Hasse diagram:



Theorem (Dilworth). Let $(P, \leq)$ be a poset with no antichain of size $> k$. Then P can be covered by $\leq k$ chains.

Proof will follow from the next theorem.

Given a poset $(P, \leq)$, define a directed graph D by $V(D) \coloneqq P$ and $\overrightarrow{uv} \in E(D)$ if $u \leq v$.

Antichain in $(P, \leq)$ $\leftrightarrow$ independent set in D

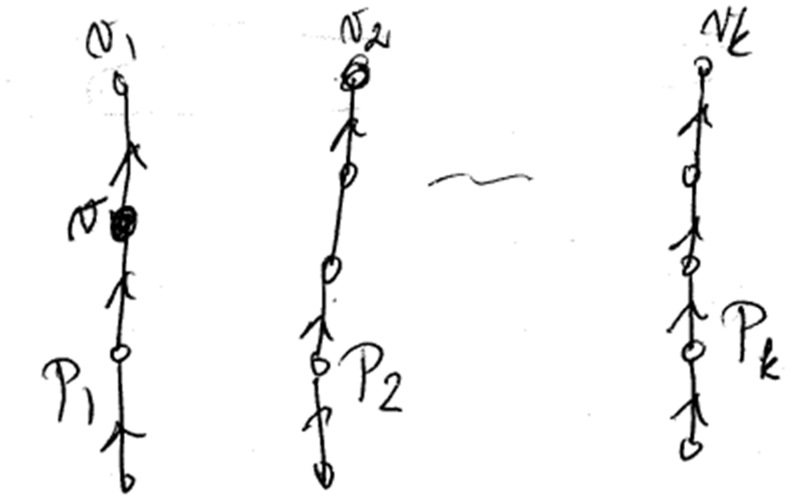
Cover by chains $\leftrightarrow$ cover by (directed) paths in D

Theorem (Gallai and Milgram). Every directed graph D has a path cover by at most $\alpha(D)$ paths.

[$\alpha(D) \coloneqq$ size of maximum independent set, path cover means paths $P_1, \dots, P_k$ such that $V(D) = V(P_1) \cup \cdots \cup V(P_k)$].

Proof. For sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of disjoint paths let $\mathcal{P}_1 < \mathcal{P}_2$ mean that:

- $|\mathcal{P}_1| < |\mathcal{P}_2|$, and
- $\forall\, P \in \mathcal{P}_1$ the terminus of P is a terminus of some path in $\mathcal{P}_2$.



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Given a poset $(P, \leq)$, define a directed graph D by $V(D) := P$ and $\overrightarrow{uv} \in E(D)$ if $u \leq v$.

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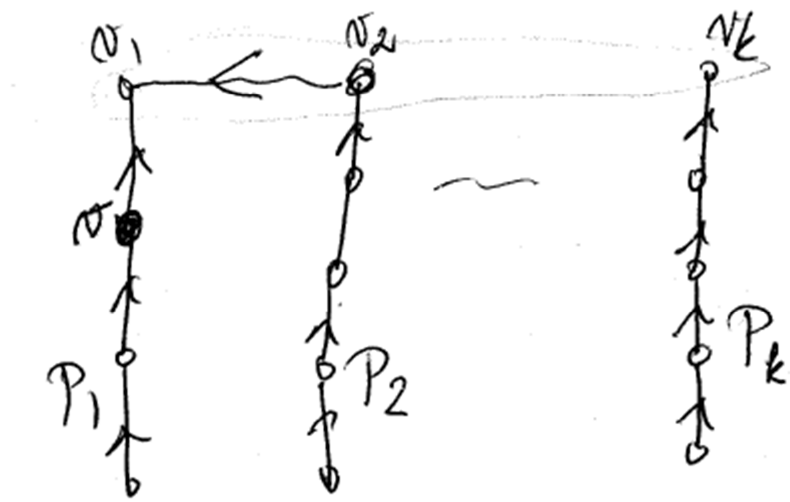


Claim. If $\mathcal{P}$ is a $<$-minimal path cover of D by disjoint paths, then there exists an independent set $\{v_p\}_{P\in\mathcal{P}}$ with $v_p \in V(P)$ for every $P \in \mathcal{P}$.

Note claim implies theorem.

Proof of Claim. Let $\mathcal{P} = \{P_1, \dots, P_k\}$. Let v_i be the terminus of P_i. If $\{v_1, \dots, v_k\}$ is independent, then it satisfies the conclusion of the claim. So WMA $\overrightarrow{v_2 v_1} \in E(D)$. Then is $P_1 \backslash v_1$ non-null.

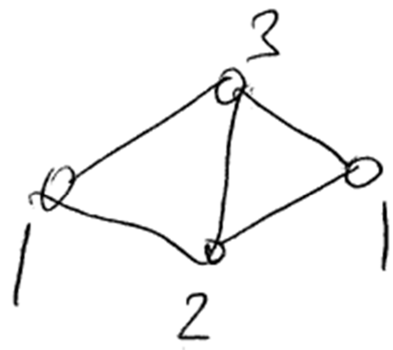
If $\mathcal{P}' \coloneqq \{P_1 \backslash v_1, P_2, \dots, P_k\}$ is $<$-minimal in $D \backslash v_1$, then done by induction. So WMA $\exists$ path cover $\mathcal{P}''$ of $D \backslash v_1$ with $\mathcal{P}'' < \mathcal{P}'$.

If v or v_2 is a terminal of a path P in $\mathcal{P}''$, then extend P by adding v_1, obtaining a path cover of D which is $< \mathcal{P}$, contrary to the minimality of $\mathcal{P}$.

So WMA v and v_2 are not termini of paths in $\mathcal{P}''$, and hence $|\mathcal{P}''| \leq k - 2$. Add the path with sole vertex v_1 to $\mathcal{P}''$ to get a path cover that is $< \mathcal{P}$, a contradiction. $\square$

Coloring

A **coloring** (vertex-coloring) of G is a function c that maps $V(G)$ to some set S in such a way that if $u \sim v$, then $c(u) \neq c(v)$. We say c is a k-**coloring** if $|S| \leq k$. The **chromatic number** of G, denoted by $\chi(G)$, is the least integer k such that G has a k-coloring.



$$\chi(G) \leq 1 \Leftrightarrow E(G) = \emptyset$$

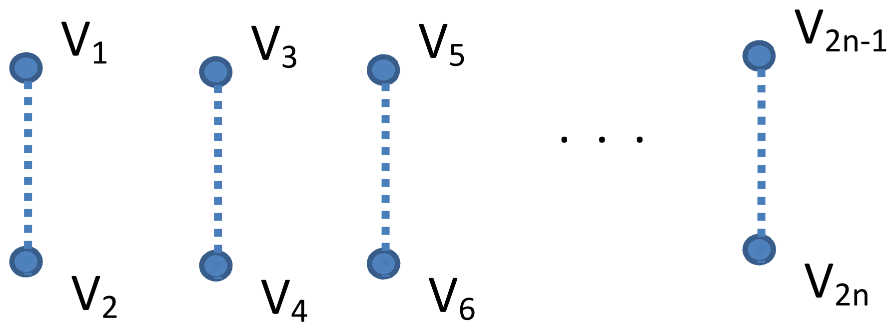
$$\chi(G) \leq 2 \Leftrightarrow G \text{ is bipartite}$$

$$\chi(G) \leq 3 \Leftrightarrow ??$$

Deciding $\chi(G) \leq 3$ is NP-hard.

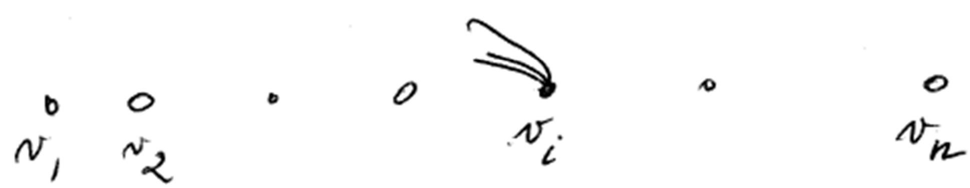
A greedy algorithm: Order the vertices $v_1, v_2, \ldots, v_n$. Having colored $v_1, \ldots, v_{i-1}$ color v_i using the least available color.

Example.



Theorem. Let $k \coloneqq \max_{H \subseteq G} \delta(H)$. Then $\chi(G) \leq k+1$.

Proof. $\delta(G) \leq k$. Let x_n be a vertex of degree $\leq k$.
$\delta(G \backslash x_n) \leq k$. Let x_{n-1} be a vertex of degree $\leq k$ in $G \backslash x_n$.
$\delta(G \backslash \{x_{n-1}, x_n\}) \leq k$. Let x_{n-2} be a vertex of degree $\leq k$ in $G\{x_{n-1}, x_n\}$.
This constructs a linear ordering $v_1, v_2, \ldots, v_n$ of $V(G)$.



The greedy algorithm will use $\leq k+1$ colors. □

Corollary. $\chi(G) \leq \Delta(G) + 1$.

[Reminder: $\Delta(G) =$ maximum degree]

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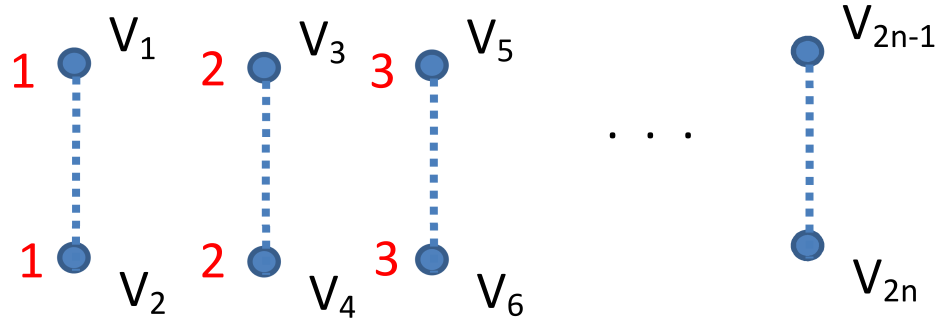

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