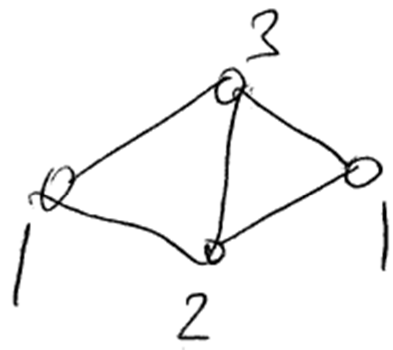
Coloring

A **coloring** (vertex-coloring) of G is a function c that maps $V(G)$ to some set S in such a way that if $u \sim v$, then $c(u) \neq c(v)$. We say c is a k-**coloring** if $|S| \leq k$. The **chromatic number** of G, denoted by $\chi(G)$, is the least integer k such that G has a k-coloring.



$\chi(G) \leq 1 \Leftrightarrow E(G) = \emptyset$

$\chi(G) \leq 2 \Leftrightarrow G$ is bipartite

Deciding $\chi(G) \leq 3$ is NP-hard.

A greedy algorithm: Order the vertices $v_1, v_2, \dots, v_n$. Having colored $v_1, \dots, v_{i-1}$ color v_i using the least available color.

Theorem. Let $k \coloneqq \max_{H \subseteq G} \delta(H)$. Then $\chi(G) \leq k + 1$.

Corollary. $\chi(G) \leq \Delta(G) + 1$.

[Reminder: $\Delta(G) =$ maximum degree]

How to compute $\max_{H \subseteq G} \delta(H)$?

Algorithm.

Input: A graph G and integer $k \geq 1$

Output: Either a subgraph H of G with $\delta(H) \geq k$, or a valid statement that no such subgraph exists

Description:

If G is null answer "no"

If $\delta(G) \geq k$, then return G

Otherwise pick $v \in V(G)$ of degree $< k$ and apply the algorithm recursively to $G \backslash v$

Theorem (Brooks) If G is connected, not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof. WMA $\Delta(G) \geq 3$. WMA G is Δ-regular.

If we can find a linear ordering

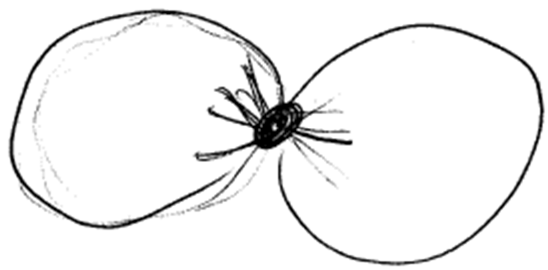
then the greedy algorithm succeeds.

Need $v_1, v_2, v_n \in V(G)$ distinct such that:

$$v_1 \sim v_n, v_2 \sim v_n, v_1 \nsim v_2 \text{ and } G\backslash\{v_1, v_2\} \text{ is connected.}$$

If G is 3-connected, then v_1, v_2, v_n exist, because $\sim$ is not transitive (because G is not complete).

If G is not 2-connected, then it can be



written as $G = G_1 \cup G_2$, where $|V(G_1) \cap V(G_2)| = 1$.

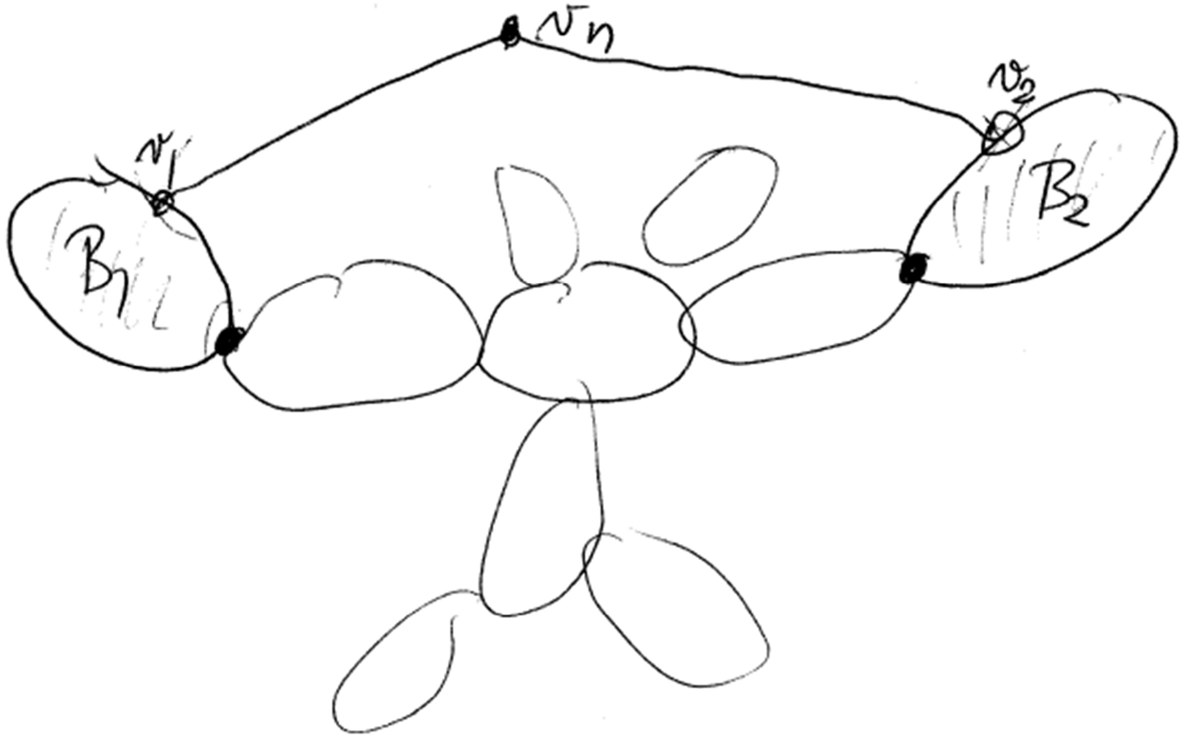
By induction

$$\chi(G_1) \leq \Delta(G_1) \leq \Delta(G)$$

$$\chi(G_2) \leq \Delta(G_2) \leq \Delta(G)$$

and hence $\chi(G) \leq \Delta(G)$, as desired.

So WMA G is 2-connected, but not 3-connected. So $\exists v_n \in V(G)$ such that $G \backslash v_n$ is connected, but not 2-connected.



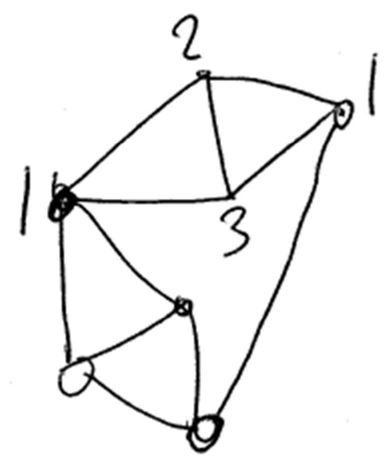
By the block structure of $G \backslash v_n$ there are two distinct end-blocks B_1, B_2. Since G is 2-connected, for $i = 1,2$ there is a neighbor v_i of v_n in B_i that is not a cut vertex of $G \backslash v_n$.

Then v_1, v_2, v_n are as desired. $\square$

$\omega(G)$ = size of a maximum clique.

clique = vertex-set of a complete subgraph. Clearly

$$\chi(G) \geq \omega(G)$$

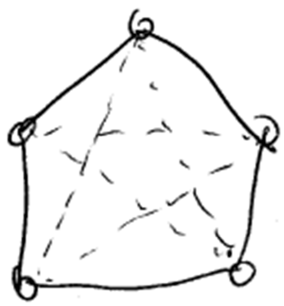


In general, $\chi(G)$ could be big, $\omega(G)$ small. In fact, there exist graphs with $\omega(G) = 2$ and $\chi(G)$ arbitrarily big. (Problem sets)

In fact, $\forall\, k, \ell\ \exists$ graph G with no cycles of length $\leq \ell$ and $\chi(G) \geq k$ (existence later).

Definition. A graph G is called **perfect** if $\chi(H) = \omega(H)$ for every **induced** subgraph H of G.

Example.



$\chi = 3$

$\omega = 2$

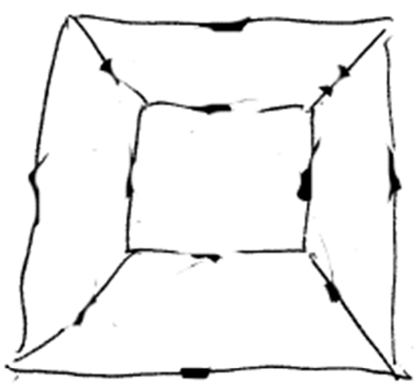
Example. Bipartite graphs are perfect (easy).

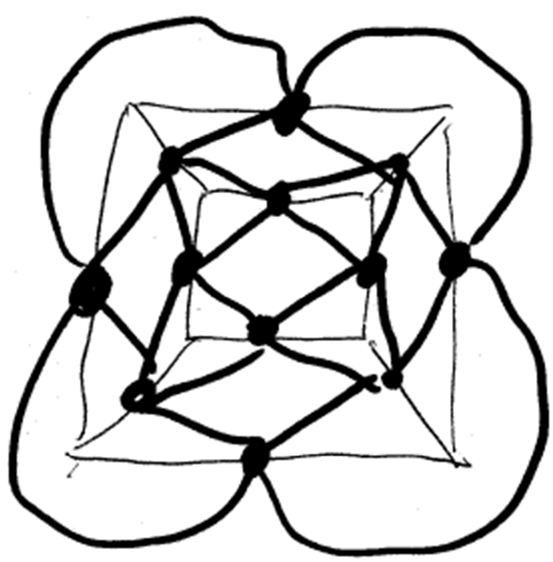
Example. Complements of bipartite graphs are perfect (exercise).

Definition. The line graph of G is a graph L defined by

$$V(L) = E(G)$$

$e, f \in V(L)$ are adjacent in L if they are adjacent in G.





Example. Line graphs of bipartite graphs are perfect (exercise).

Example. Complements of line graphs of bipartite graphs are perfect (exercise).

Sample argument: $L =$ Line graph (G), G bipartite.

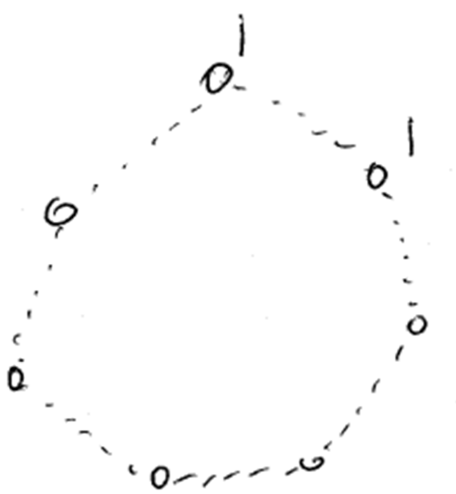
Enough to show $\chi(L) = \omega(L)$:

$\omega(L) = \Delta(G)$

$\chi(L) =$ edge-chromatic number of $G = \chi'(G)$.

Example. Odd cycles of length ≥ 5 are not perfect.

Example. Complements of odd cycles of length ≥ 5 are not perfect .



$\chi(C_{2k+1}^c) = k + 1$

$\omega(C_{2k+1}^c) = k$

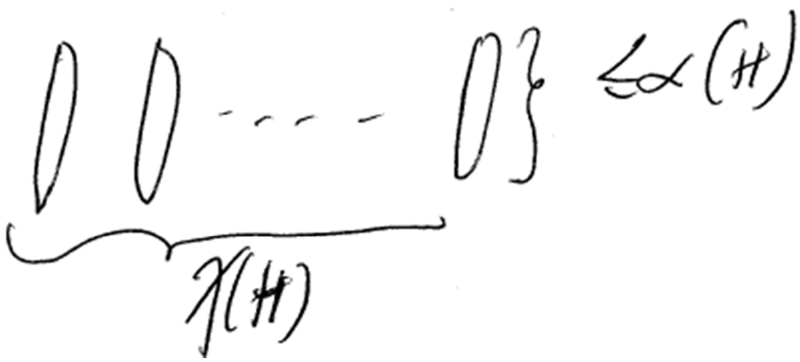
$\alpha(G)$ = size of maximum independent set

$\omega(G)$ = size of maximum clique

$$\omega(G) = \alpha(G^c)$$

Lemma. $|V(H)| \leq \chi(H)\alpha(H)$ for every graph H.

Proof.



G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

The weak perfect graph theorem (Lovász)
G is perfect $\Leftrightarrow$ G^c is perfect.

This is a deep theorem. A proof will follow from

Theorem (Lovász) A graph G is perfect

$$\Updownarrow$$

$|V(H)| \leq \alpha(H)\omega(H)$ for every induced subgraph of G.

Proof. $\Downarrow$: Let G be perfect, and let H be an induced subgraph.

$$|V(H)| \leq \chi(H)\alpha(H) = \omega(H)\alpha(H)$$

by the lemma and perfection of G.

$\Uparrow$: Follows from next thm.

Theorem If G is minimally imperfect, then

$$|V(G)| = \alpha(G)\omega(G) + 1$$

Def. G is minimally imperfect if G is not perfect and $G\backslash v$ is perfect for every $v \in V(G)$.

Observations. If G is minimally imperfect and $\omega := \omega(G)$, then

(a) $\chi(G) = \omega + 1$, and
(b) if $S \subseteq V(G)$ is independent, then $\omega(G\backslash S) = \omega$.

Proof. (a) $\chi(G) > \omega$, because $\chi(H) = \omega(H)$ for every proper induced subgraph H of G.

$$\chi(G) \leq \chi(G\backslash v) + 1 = \omega(G\backslash v) + 1 \leq \omega + 1$$

(b) If $\chi(G\backslash S) = \omega(G\backslash S) < \omega$, then color $G\backslash S$ using $\leq \omega - 1$ colors and add S to get an ω-coloring of G, contrary to (a).

Theorem If G is minimally imperfect, then

$$|V(G)| = \alpha(G)\omega(G) + 1$$

Proof. $\leq$: easy and not needed (exercise).

$\geq$: Let $n := |V(G)|$, $\alpha := \alpha(G)$ and $\omega := \omega(G)$. We must show that $n \geq \alpha\omega + 1$. We will do so by constructing $\alpha\omega + 1$ linearly independent vectors in $\mathbb{R}^n$.

Let $S_0 := \{v_1, v_2, \dots, v_\alpha\}$ be a maximum independent set.
For $i = 1,2, \dots, \alpha$ pick an ω-coloring of $G \backslash v_i$:

$$S_{(i-1)\omega+1}, S_{(i-1)\omega+2}, \dots, S_{(i-1)\omega+\omega} \qquad (*)$$

$G \backslash v_1$ has ω-coloring $S_1, S_2, \dots, S_\omega$

$G \backslash v_2$ has ω-coloring $S_{\omega+1}, S_{\omega+2}, \dots, S_{2\omega}$

$G \backslash v_i$ has ω-coloring $S_{(i-1)\omega+1}, S_{(i-1)\omega+2}, \dots, S_{(i-1)\omega+\omega}$

$G \backslash v_\alpha$ has ω-coloring $S_{(\alpha-1)\omega+1}, S_{(\alpha-1)\omega+2}, \dots, S_{\alpha\omega}$

Thus we have $\alpha\omega + 1$ independent sets $S_0, S_1, \dots, S_{\alpha\omega}$.

Claim. Each maximum clique is disjoint from exactly one S_i.

Pf. Let Q be a maximum clique. If $v_i \notin Q$, then Q is an ω-clique in $G \backslash v_i$. Since $(*)$ is an ω-coloring of $G \backslash v_i$, Q intersects all those independent sets.

If $S_0 \cap Q = \emptyset$, then S_0 is the unique S_i disjoint from Q.

Otherwise $v_i \in Q$ for some i and $Q - \{v_i\}$ is an $(\omega - 1)$-clique in $G \backslash v_i$ and so Q intersects all but one of the sets in $(*)$. This proves the claim.

WMA $V(G) = \{1, \dots, n\}$. I claim $\mathbb{1}_{S_0}, \mathbb{1}_{S_1}, \dots, \mathbb{1}_{S_{\alpha\omega}}$ are linearly independent. Note that $\omega(G \backslash S_i) = \omega$ for all $i = 0,1, \dots, \alpha\omega$ by Observation. Thus there exists a maximum clique Q_i disjoint from S_i. We have

$$\mathbb{1}_{S_i} \cdot \mathbb{1}_{Q_j} = \left|S_i \cap Q_j\right| = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases}$$

Suppose $\sum \lambda_i \mathbb{1}_{S_i} = 0$. Then

$$0 = \left(\sum_{i=0}^{\alpha\omega} \lambda_i \mathbb{1}_{S_i}\right) \cdot \mathbb{1}_{Q_j} = \sum_{i=0}^{\alpha\omega} \lambda_i \left(\mathbb{1}_{S_i} \cdot \mathbb{1}_{Q_j}\right) = \left(\sum_{i=0}^{\alpha\omega} \lambda_i\right) - \lambda_j$$

for every $j = 0,1, \dots, \alpha\omega$. Thus $\lambda_0 = \lambda_1 = \cdots = \lambda_{\alpha\omega} = (\sum_{i=0}^{\alpha\omega} \lambda_i)$, and so they are all 0. $\square$

The Strong Perfect Graph Theorem

A **hole** in a graph is an induced cycle of length at least 4. An **antihole** is the complement of a hole.

Strong Perfect Graph Theorem. A graph is perfect if and only if it has no odd hole and no odd antihole.

Implies the Weak Perfect Graph Theorem.