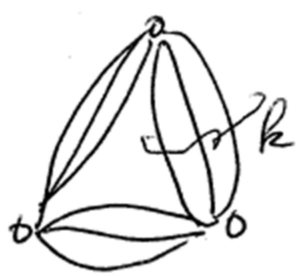
An **edge-coloring** of G is a mapping $c: E(G) \to S$, where S is some set, such that $c(e) \neq c(f)$ for every two adjacent edges e, f. It is a k-edge-coloring if $|S| \leq k$. The **edge-chromatic number** or **chromatic index** of G, denoted by $\chi'(G)$, is the least integer k such that G has a k-edge-coloring.

Clearly $\Delta(G) \leq \chi'(G)$ and $\chi'(G) = \chi\big(L(G)\big)$.

Observation. $\chi'(G) = \chi\big(L(G)\big) \leq \Delta\big(L(G)\big) + 1 \leq 2\Delta(G) - 1$

Theorem (Vizing) For every simple graph $\chi'(G) \leq \Delta(G) + 1$.

Example.

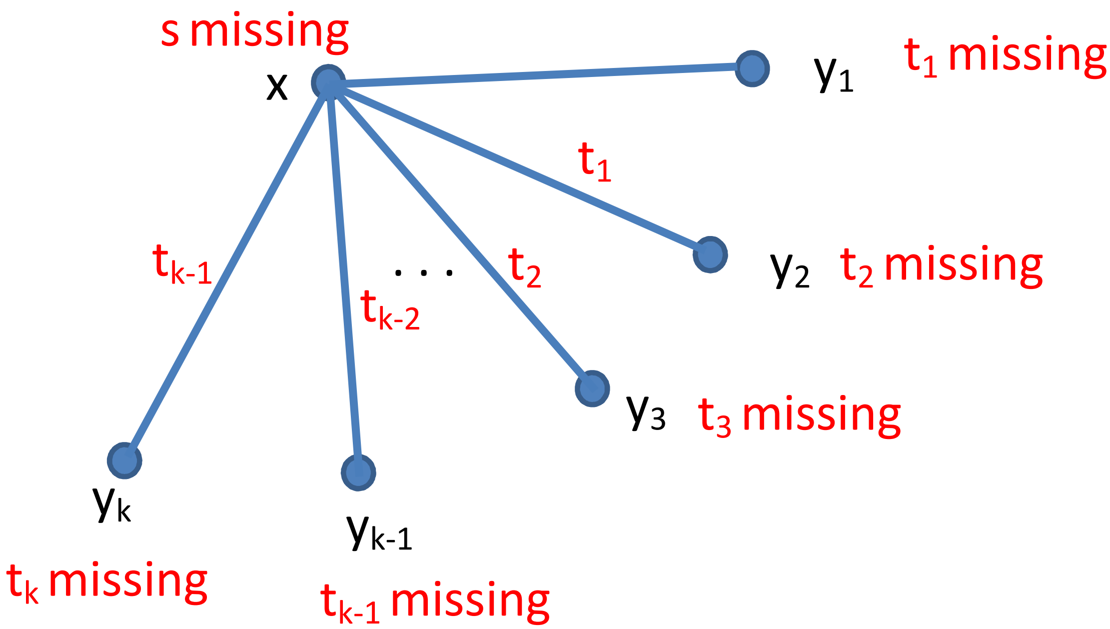


$|E(G)| = 3k$

$\Delta(G) = 2k$

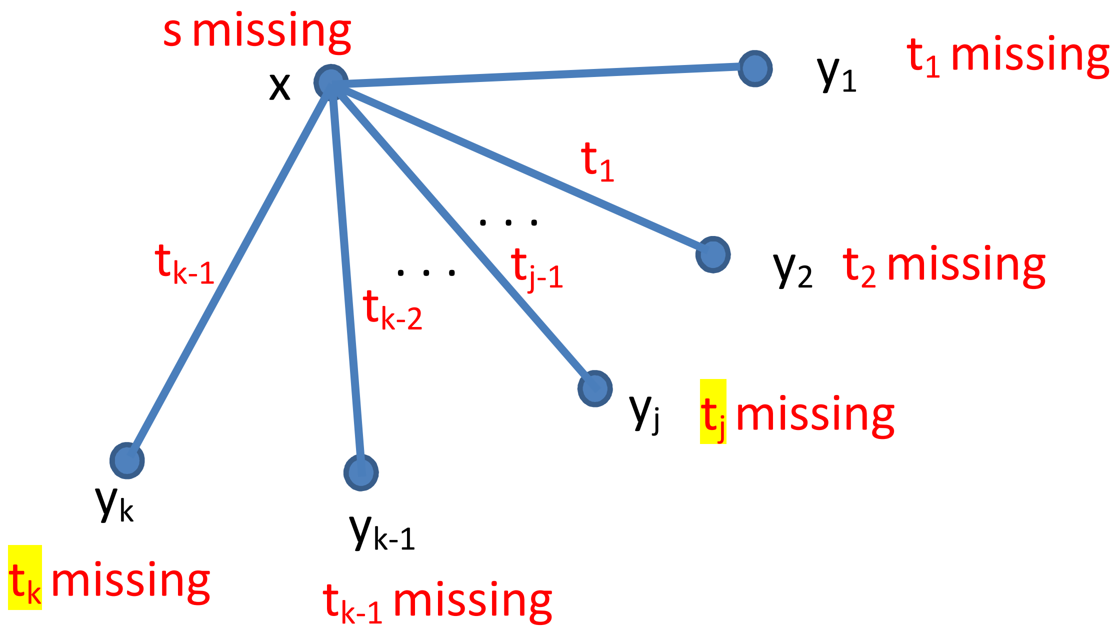
$\chi'(G) = 3k$

Proof. By induction on $|E(G)|$. By induction we can color all but one edge of G using $\Delta(G) + 1$ colors. Let xy_1 be the uncolored edge. For every $v \in V(G)$ there is a "missing color" at v; that is, a color not used by any edge incident with v.



Construct this for as long as $t_1, \dots, t_k$ are pairwise distinct and the edges xy_i exist.

Case 1. t_k is missing at x. Color xy_i using t_i for $i = 1,2, \dots, k$.



Construct this for as long as $t_1, \dots, t_k$ are pairwise distinct and the edges xy_i exist.

Case 1. t_k is missing at x. Color xy_i using t_i for $i = 1,2, \dots, k$.

Case 2. $t_k = t_j$ for some $j = 1,2, \dots, k-1$.

Let H be the subgraph of G consisting of edges colored s or t_k.

Notice that if we swap s and t_k in any component of H we get a valid edge-coloring. Either (x and y_j) or (x and y_k) are not in the same component of H.

Case 2a. x, y_j are not in the same component of H

Swap s, $t_k = t_j$ in the component of H containing y_j. Color xy_j using s, color xy_i using t_i for $i = 1,2, \dots, j-1$.

Case 2b. Analogous (replace j by k). □

A communication model

Input alphabet Σ, output alphabet Σ_{out}

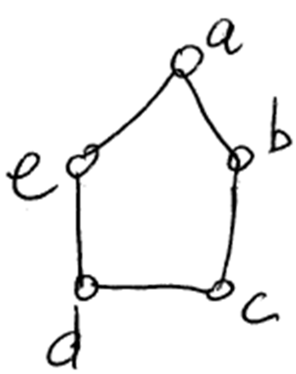
On input a we might receive a_1 or a_2 or a_3 or $\cdots$

On input b we might receive b_1 or b_2 or b_3 or $\cdots$

On input c we might receive c_1 or c_2 or c_3 or $\cdots$

a, b are confoundable if $a_j = b_i$ for some i, j.

Example. $\sum = \{a, b, c, d, e\}$



The graph indicates confoundable pairs.

Two sequences $(x_1, \dots, x_t), (y_1, \dots, y_t)$ of elements of Σ are **confoundable** if $\forall i = 1,2, \dots, t$ either $x_i = y_i$ or x_i, y_i are confoundable.

Objective. A set of pairwise unconfoundable sequences of length t

Example 1. Any sequence of a's and c's of length t. That is a family of 2^t pairwise unconfoundable sequences of length t.

Example 2. A bigger family. Notice that

$$aa, bd, cb, de, ec$$

are pairwise unconfoundable. Take any sequence of these of length $t/2$. That will give a collection of pairwise unconfoundable words of size $5^{t/2} = (2^{\log 5})^{\frac{t}{2}} = 2^{(1/2 \log 5)t}$.

Fekete's lemma. If $(a_t)_{t\geq 1}$ is a sequence of positive real numbers satisfying

$$a_{s+t} \geq a_s + a_t$$

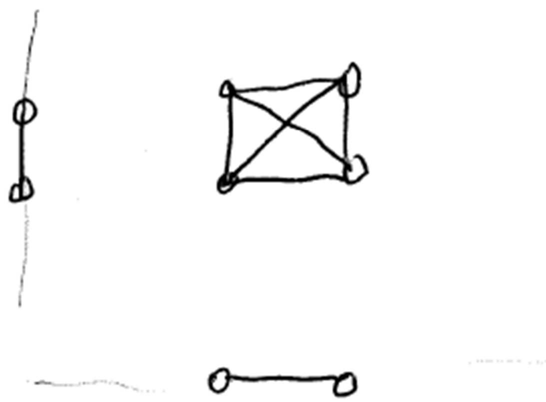
then $\lim_{t\to\infty} \frac{1}{t} a_t$ exists and is equal to $\sup_{t\geq 1} \frac{1}{t} a_t$.

Definition. For graphs K, L we define their product $K \boxtimes L$ by $V(K \boxtimes L) = V(K) \times V(L)$ and

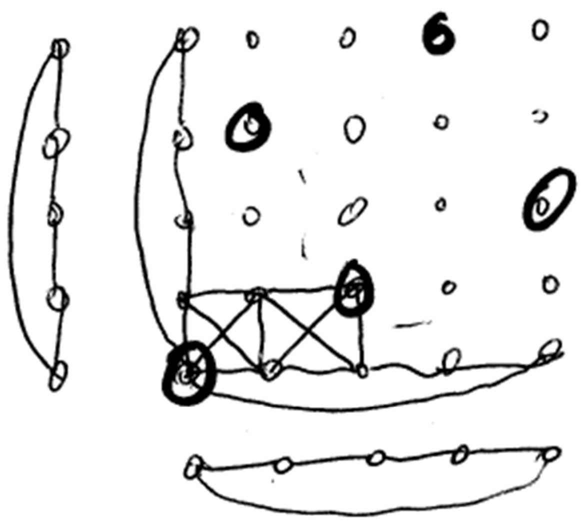
$(k_1, \ell_1) \sim (k_2, \ell_2)$ if

- $(k_1, \ell_1) \neq (k_2, \ell_2)$ and
- $k_1 = k_2$ or $k_1 \sim k_2$ in K and
- $\ell_1 = \ell_2$ or $\ell_1 \sim \ell_2$ in L

Example. $K_2 \boxtimes K_2$



Example. $C_5 \boxtimes C_5$



Let $\gamma(G) \coloneqq \chi(G^c) = \min \#$ of cliques covering the vertices of G.

Observations: (1) $\alpha(G_1 \boxtimes G_2) \geq \alpha(G_1)\alpha(G_2)$

(2) $\alpha(C_5 \boxtimes C_5) = 5$

(3) $\gamma(G_1)\gamma(G_2) \geq \gamma(G_1 \boxtimes G_2)$

Proof of (3). Let $K_1, \ldots, K_r$ be a cover by cliques of G_1.

Let $L_1, \ldots, L_s$ be a cover by cliques of G_2.

Then $\{K_i \times L_j : 1 \leq i \leq r, 1 \leq j \leq s\}$ is a cover of $G_1 \boxtimes G_2$ by rs cliques, as desired.

Let $G^t \coloneqq G \boxtimes G \boxtimes \cdots \boxtimes G$ (t times). The **Shannon capacity** of G is defined as

$$\lim_{t\to\infty} \frac{1}{t} \log \alpha\,(G^t)$$

By (1), $\alpha(G^{s+t}) \geq \alpha(G^s)\alpha(G^t)$

$$\log \alpha\,(G^{s+t}) \geq \log \alpha\,(G^s) + \log \alpha\,(G^t)$$

By Fekete's lemma $\lim_{t\to\infty} \frac{1}{t} \log \alpha\,(G^t)$ exists and is equal to $\sup_{t\geq 1} \frac{1}{t} \log \alpha\,(G^t)$.

If $\alpha(G) = \gamma(G)$, then

$$(\gamma(G))^t \geq \gamma(G^t) \geq \alpha(G^t) \geq (\alpha(G))^t$$

and so equality holds throughout. Thus the Shannon capacity of G is $\log \alpha(G)$.

What are the minimal graphs that do not satisfy $\alpha(G) = \gamma(G)$? Those are precisely minimally imperfect graphs.

Theorem (Lovász) The Shannon capacity of C_5 is $(\log 5)/2$.

We do not know the Shannon capacity of C_7 or other odd holes or odd antiholes.