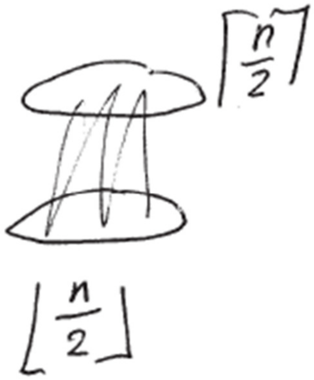
Extremal problems

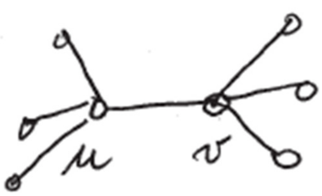
How many edges can a triangle-free graph on n vertices have?



$$\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \approx \frac{n^2}{4} \text{ edges}$$

Theorem (Matel) If G has no triangle and $n = |V(G)|$, then $|E(G)| \leq \frac{n^2}{4}$.

Proof. Let $uv \in E(G)$.



Neighbors disjoint $\Rightarrow$

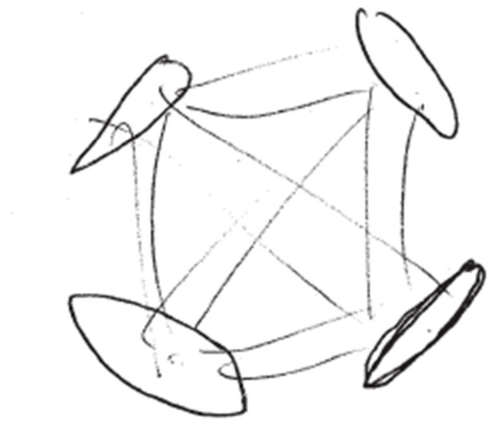
$\deg(u) + \deg(v) \leq n$

$$n|E(G)| \geq \sum_{uv \in E(G)} (\deg(u) + \deg(v)) =$$

$$= \sum_{w \in V(G)} \deg^2(w) \geq \frac{1}{n} \left(\sum_{w \in V(G)} \deg(w) \right)^2 = \frac{4}{n} |E(G)|^2$$

$\Longrightarrow \ |E(G)| \leq \frac{n^2}{4}$ $\square$

Fix r. What is the maximum number of edges a graph with no K_r subgraph can have?



An extremal graph $T_{r-1}(n)$, "the Turán graph", is the complete multipartite graph with $r-1$ parts that are as close as possible to each other in size. It has all edges between different parts. Each part has size $\left\lfloor \frac{n}{r-1} \right\rfloor$ or $\left\lceil \frac{n}{r-1} \right\rceil$.

If $r-1$ divides n, then

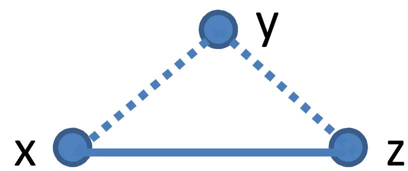
$$\left|E\left(T_{r-1}(n)\right)\right| = \binom{r-1}{2}\left(\frac{n}{r-1}\right)^2 = \frac{(r-1)(r-2)}{2}\frac{n^2}{(r-1)^2} = \frac{1}{2}\frac{r-2}{r-1}n^2$$

Theorem (Turán) If G is a graph on n vertices with no K_r-subgraph, then

$$|E(G)| \leq |E(T_{r-1}(n))|$$

with equality if and only if G is isomorphic to $T_{r-1}(n)$.

Proof #1. Let G have no K_r-subgraph and maximum number of edges. We claim G is complete multipartite. If not, then $\nsim$ is not transitive, and so there exist x, y, z:



If $\deg(x) > \deg(y)$, then deleting y and cloning x produces a graph with no K_r-subgraph and $> |E(G)|$ edges, a contradiction. So WMA $\deg(x) \leq \deg(y)$ and similarly $\deg(z) \leq \deg(y)$. Now deleting both x and z and cloning y twice produces a graph with no K_r-subgraph and $> |E(G)|$ edges, a contradiction. This proves our claim that G is complete multipartite.

The maximality of G implies that G is isomorphic to $T_k(n)$ for some k. Since G has no K_r-subgraph it follows that $k \leq r - 1$, and by comparing the degrees of vertices in $T_k(n)$ and $T_{r-1}(n)$ we conclude that the maximality of G implies that $k = r - 1$.

Lemma.

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1},$$

with equality if and only if G is a disjoint union of cliques.

Proof. Let $<$ be a linear ordering of $V(G)$.

$$I(<) := \{v \in V(G) : \forall\, w \quad vw \in E(G) \Rightarrow v < w\}$$

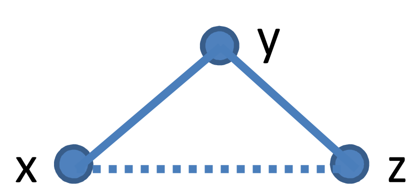
Then $I(<)$ is independent. Now choose $<$ uniformly at random.

$$X(<) \coloneqq |I(<)| = \sum_{v \in V(G)} \mathbb{1}_{[v \in I]}$$

$$EX = \sum_{v \in V(G)} E\, \mathbb{1}_{[v \in I]} = \sum_{v \in V(G)} P[v \in I] = \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}$$

There exists $<$ such that $|I(<)| \geq EX$; then $I(<)$ is an independent set of size $\geq \sum_{v \in V(G)} \frac{1}{\deg(v)+1}$, as required.

Assume now G is not a union of cliques. We will prove inequality is strict. $\Rightarrow \exists\ x, y, z$:



Define

$<_1 : \quad x, y, z, \dots \dots$

$<_2 : \quad x, z, y, \dots \dots$

Then $I(<_1) = \{x, \cdots\}$ and $I(<_2) = \{x, z, \cdots\}$, and so $|I(<_1)| < |I(<_2)|$. Thus X is not constant, and hence there exists a linear ordering $<$ such that $X(<) > EX$, and so

$\alpha(G) > \sum_{v \in V(G)} \frac{1}{\deg(v)+1}$, as desired. $\square$

Corollary. If G has n vertices and e edges, then

$$\alpha(G) \geq \frac{n^2}{2e+n}$$

Proof. If $a_1 + \cdots + a_n =$ const, then $\sum_{i=1}^{n} \frac{1}{a_i}$ is minimized when the a_i's are equal. Thus

$$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v)+1} \geq \sum_{v \in V(G)} \frac{1}{\frac{2e}{n}+1} = \frac{n}{\frac{2e}{n}+1} = \frac{n^2}{2e+n}$$

$\square$

Theorem 2. Let H be a graph on n vertices such that $|E(H)| = |E\left(T_{r-1}^c(n)\right)|$. Then $\alpha(H) \geq r-1$ with equality if and only if $H \cong T_{r-1}^c(n)$.

Proof of Turán's theorem, assuming Theorem 2. Let G have n vertices and no K_r subgraph. WMA

$$|E(G)| \geq \left|E\left(T_{r-1}(n)\right)\right|,$$

for otherwise we are done. Let H be a spanning supergraph of G^c with $|E\left(T_{r-1}^c(n)\right)|$ edges. Then $\alpha(H) \leq r-1$. By Theorem 2 $\alpha(H) \geq r-1$, and so $H \cong T_{r-1}^c(n)$. Thus G has a spanning subgraph isomorphic to $T_{r-1}(n)$, and hence $G \cong T_{r-1}(n)$, because adding any edge to $T_{r-1}(n)$ creates a K_r-subgraph. □

Proof of Theorem 2. By the lemma

$$\alpha(H) \geq \sum_{v \in V(H)} \frac{1}{\deg_H(v)+1} \geq \sum_{v \in T_{r-1}^c(n)} \frac{1}{\deg_{T_{r-1}^c}(v)+1} = r-1$$

The second inequality holds because the previous expression is minimized when the degrees are as close to each other as possible.

Equality in 1st inequality $\Leftrightarrow$ H is a union of cliques.

Equality in 2nd inequality $\Leftrightarrow$ degrees are as close to each other as possible.

$\Rightarrow$ Equality in Theorem 2 if and only $H \cong T_{r-1}^c(n)$. □