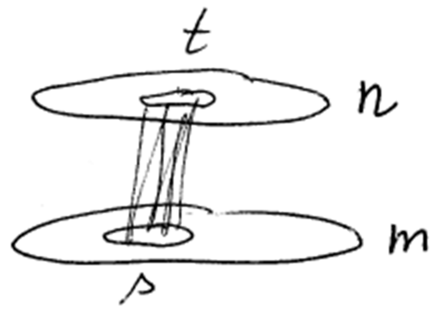
The problem of Zarankiewicz

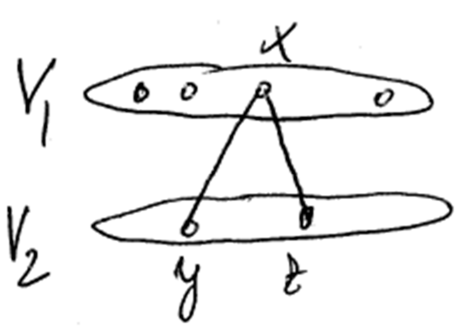
The problem: What is the maximum number of edges a bipartite graph G with no $K_{s,t}$ subgraph can have?



Let $z(m,n,s,t)$ denote the maximum number of edges in G with no such $K_{s,t}$ subgraph.

Theorem. $z(n,n,2,2) \leq \frac{n(1+\sqrt{4n-3})}{2}$ and equality holds for infinitely many n.

Proof. Let $z = \frac{n(1+\sqrt{4n-3})}{2}$; then $(z-n)z = n^2(n-1)$. Suppose G is bipartite with n vertices in each class, no C_4 and more than z edges.



A *vee* is a pair $(x,\{y,z\})$, where $x \in V_1$ and y,z are distinct neighbors of x.

Let $d_1, d_2, \dots, d_n$ be the degrees of the vertices in V_1.

There are $\sum_{i=1}^{n}\binom{d_i}{2}$ vees, but no two have the same feet. Thus

$$\binom{n}{2} \geq \sum_{i=1}^{n}\binom{d_i}{2} = \frac{1}{2}\sum_{i=1}^{n} d_i^2 - \frac{1}{2}\sum_{i=1}^{n} d_i \geq \frac{1}{2n}\left(\sum d_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} d_i =$$

$$= \frac{1}{2n}|E(G)|^2 - \frac{1}{2}|E(G)| = \frac{|E(G)|(|E(G)| - n)}{2n} > \frac{z(z-n)}{2n} =$$

$$= \frac{n^2(n-1)}{2n} = \binom{n}{2},$$

a contradiction.

In order to get equality we need:

- $\forall$ distinct $y, z \in V_2$ there is a vee with feet y, z
- $d_1 = d_2 = \cdots = d_n$

Since $\binom{n}{2} = \sum_{i=1}^{n}\binom{d_i}{2} = n\binom{d_i}{2}$ we have

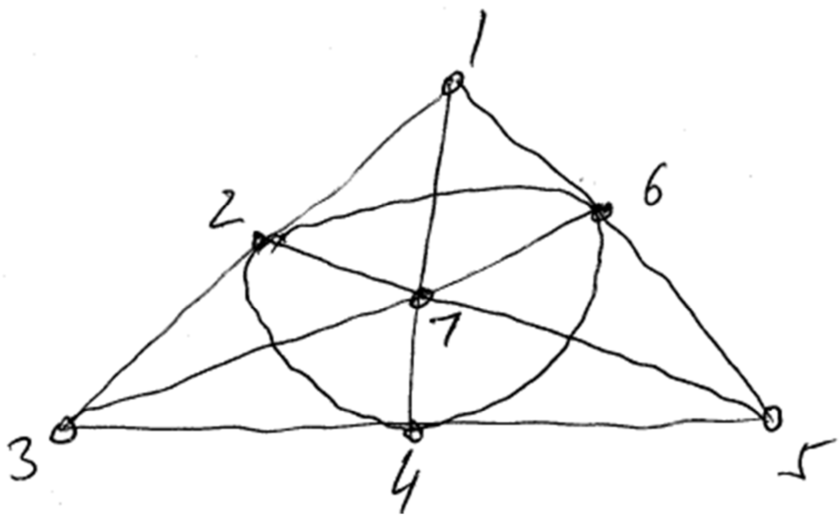
$$n = d_i^2 - d_i + 1 = (d_i - 1)^2 + (d_i - 1) + 1$$

Call elements of V_1 points and elements of V_2 lines and say a point x belongs to a line $L \in V_2$ if they are adjacent in G. Thus we arrive at

Definition. A **projective plane** of order q is a pair $(X, \mathcal{L})$, where X is a finite set and $\mathcal{L}$ is a set of subsets of X such that

(1) $|X| = q^2 + q + 1$
(2) $\mathcal{L}$ is a set of $(q+1)$-element subsets of X
(3) $\forall\, x, y \in X$ distinct there is a **unique** $L \in \mathcal{L}$ such that $x, y \in L$.

Example. $q = 2$



Theorem. (2') Every point is in exactly $q + 1$ lines.

(1') There are $q^2 + q + 1$ lines.

(3') Every two lines meet in exactly one point.

Proof. (2') Let $p \in X$. There are $q(q+1)$ other points. Each line through p has q points other than p and so there are $q + 1$ lines through p.

(1') Let $N = |\{(p, L) : p \in L\}|$.

$$N = \sum_{p \in X} (\#\text{lines through } p) = (q^2 + q + 1)(q + 1)$$

$$N = \sum_{L \in \mathcal{L}} |L| = |\mathcal{L}|(\mathrm{q} + 1)$$

(3') Let $L_1, L_2 \in \mathcal{L}$ and fix $p \in L_1 - L_2$. For every point $x \in L_2$ there is a line containing x, p. That is $q + 1$ distinct lines, and so those are all the lines containing p. But L_1 is one of those lines, and so it contains some $x \in L_2$. Thus L_1, L_2 intersect. □

Theorem. A projective plane of order q exists whenever q is a prime power.

Coming back to the case of equality. If a projective plane of order q exists, then equality holds for some graph G when $n = q^2 + q + 1$:

$V_1 =$ points, $V_2 =$ lines, adjacency is containment.

The graph satisfies $|E(G)| = z$, because equality holds throughout in the previous calculation. □

Lemma. Let V be an n-dimensional vector space over a finite field F with q elements. Then there are

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$$

ordered k-tuples of linearly independent elements of V and there are

$$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}$$

k-dimensional subspaces of V.

Theorem. A projective plane of order q exists whenever q is a prime power.

Proof. Let F be the finite field with q elements and let V be the 3-dimensional vector space over F.

Points = all 1-dimensional subspaces of V

Lines = all 2-dimensional subspaces of V

The number of points is

$$\frac{q^3 - 1}{q - 1} = q^2 + q + 1$$

every line contains

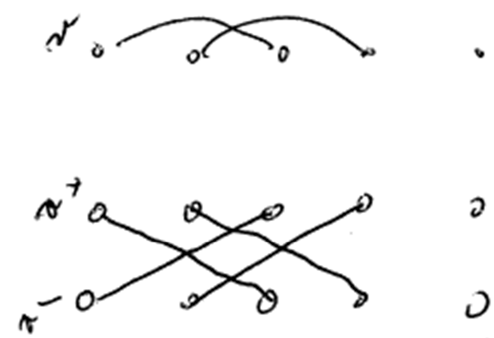
$$\frac{q^2 - 1}{q - 1} = q + 1$$

points, and every two distinct points belong to a unique line.

Corollary. If G is a (not necessarily bipartite) graph with no C_4 subgraph, then

$$|E(G)| \leq \frac{n}{4}\left(1+\sqrt{4n-3}\right) = (1+o(1))\frac{1}{2}n^{3/2}$$

Proof. Construct a bipartite graph H



H has no C_4 and so

$$|E(G)| \leq \frac{1}{2}|E(H)| \leq \frac{1}{2}\frac{n(1+\sqrt{4n-3})}{2}$$

Definition. A **subdivision** of a graph G is any graph obtained from G by replacing edges by internally disjoint paths with the same ends.

A $\boldsymbol{K_r}$**-subdivision** in G is a subgraph of G isomorphic to a subdivision of K_r.

Theorem. For every r there exists c such that every graph with average degree $\geq c$ has a K_r subdivision.

Remark. If $|V(G)| = n$, then

$$|E(G)| \geq \frac{c}{2}n \Rightarrow K_r \text{ subdivision}$$

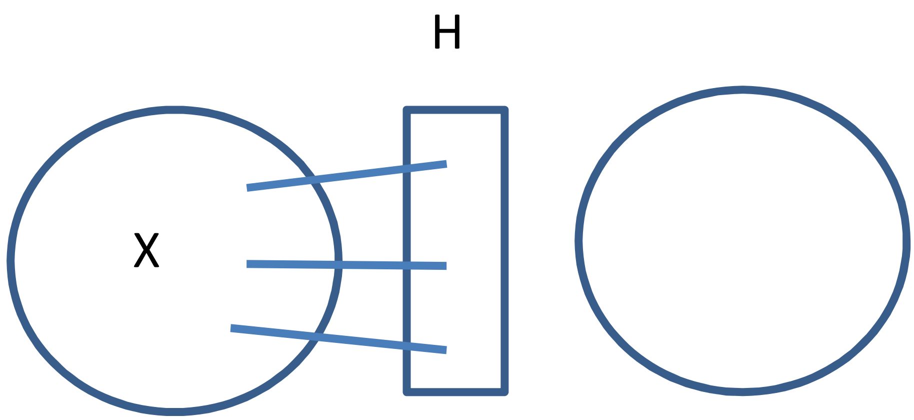
$$|E(G)| \geq c'n^2 \Rightarrow K_r \text{ subgraph}$$

The theorem will follow from

Theorem. Let $p \geq 3$. For all $m = p, p+1, \ldots, \binom{p}{2}$ every connectd graph G of average degree $\geq 2^m$ has an $\widetilde{H}$-subdivision for some graph $\widetilde{H}$ on p vertices and m edges.

Proof. Induction on m. For $m = p$, if G has average degree $\geq 2^p$, then it has a subgraph of minimum degree $\geq 2^{p-1} + 1$ (keep deleting vertices of degree $\leq 2^{p-1}$). But $2^{p-1} + 1 \geq p + 1$, and hence G has a cycle on $\geq p$ vertices.

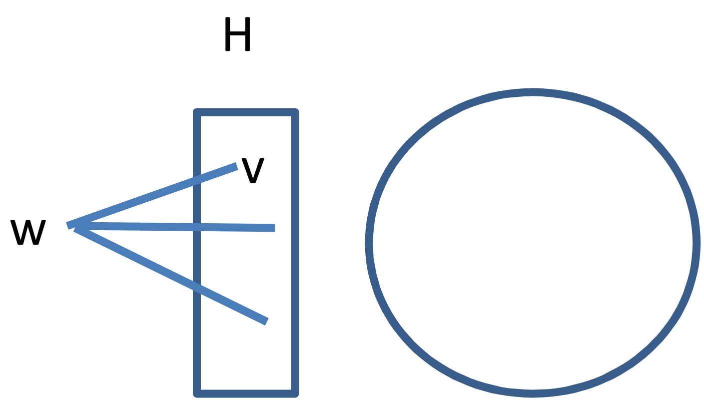
Suppose $m \geq p + 1$ and the theorem holds for smaller m. Take a maximal set $X \subseteq V(G)$ such that $G[X]$ is connected and $\text{avdeg}(G/X) \geq 2^m$. Let $H := G[N(X)]$.



Claim. H has min degree $\geq 2^{m-1}$.

Proof. Suppose $\deg_H(v) < 2^{m-1}$. Let $X' := X \cup \{v\}$.

G/X

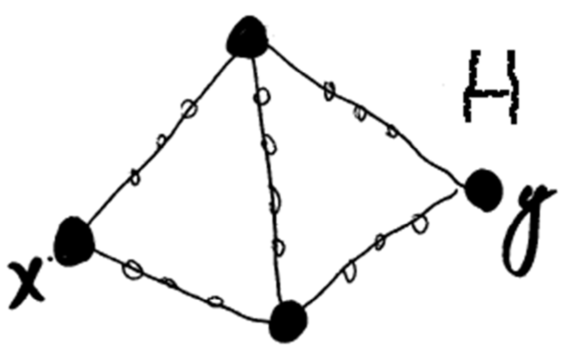


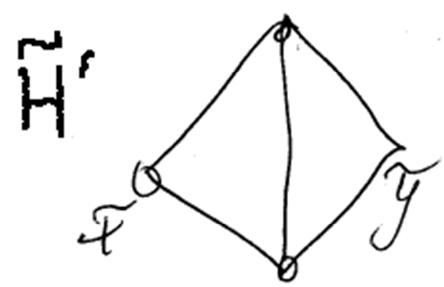
G/X' is obtained from G/X by contracting wv. In the process we lose 1 vertex, the edge wv and all edges of H incident with v and only those edges. So we lose $\leq 2^{m-1}$ edges.

avdeg$(G/X) \geq 2^m \Leftrightarrow$ edge-density $(G/X) \geq 2^{m-1}$

The edge-density of G/X' is $\geq 2^{m-1}$, because we lost 1 vertex and $\leq 2^{m-1}$ edges, contrary to the choice of X. This proves the claim.

By induction H has an $\widetilde{H'}$-subdivision, where $\widetilde{H'}$ has p vertices and $m-1$ edges.





Let $\tilde{x}, \tilde{y}$ be non-adjacent vertices of $\widetilde{H'}$, and let x, y be the corresponding vertices of H. Join x, y by a path through X. □