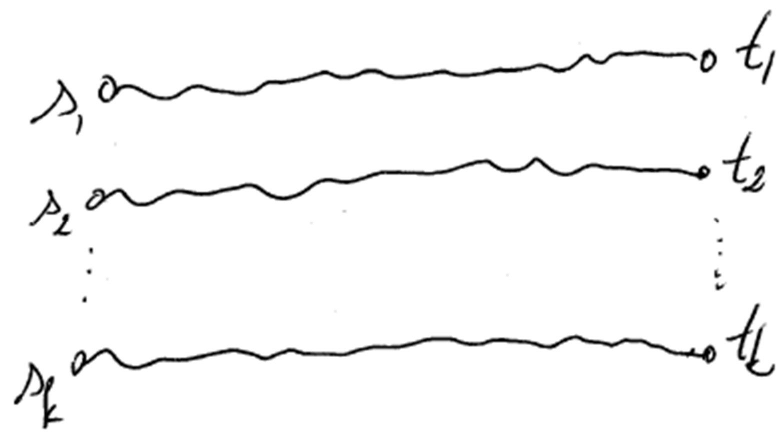
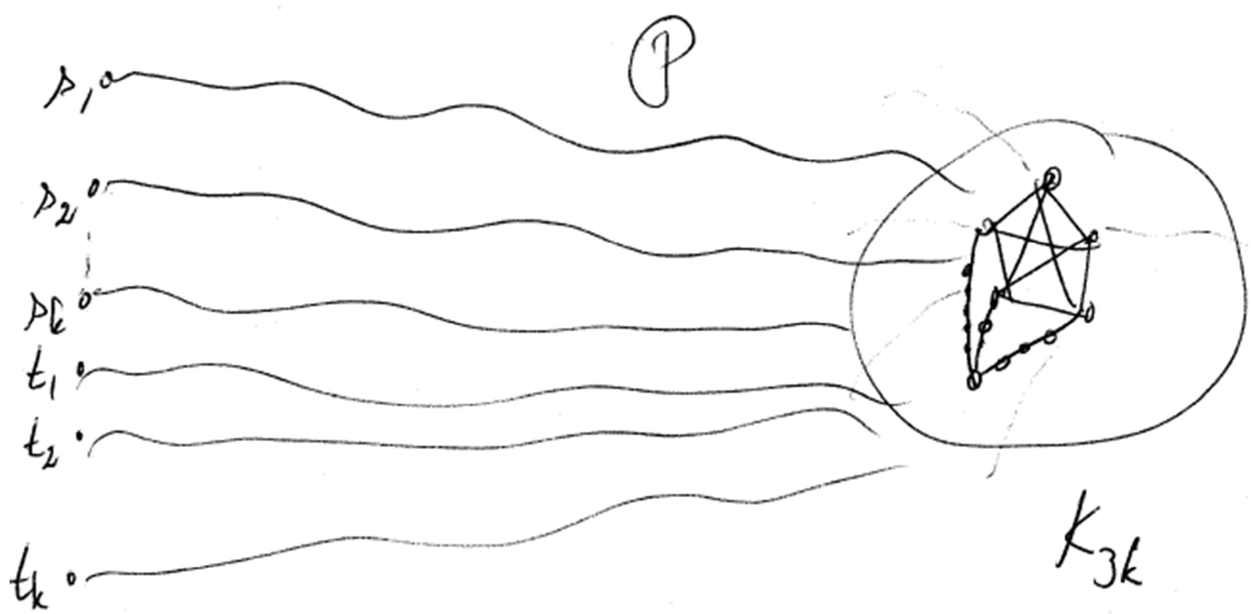
An application

Def. A graph G is k-**linked** if for every $2k$-tuple $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of pairwise distinct vertices of G there exist k disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i has ends s_i and t_i.



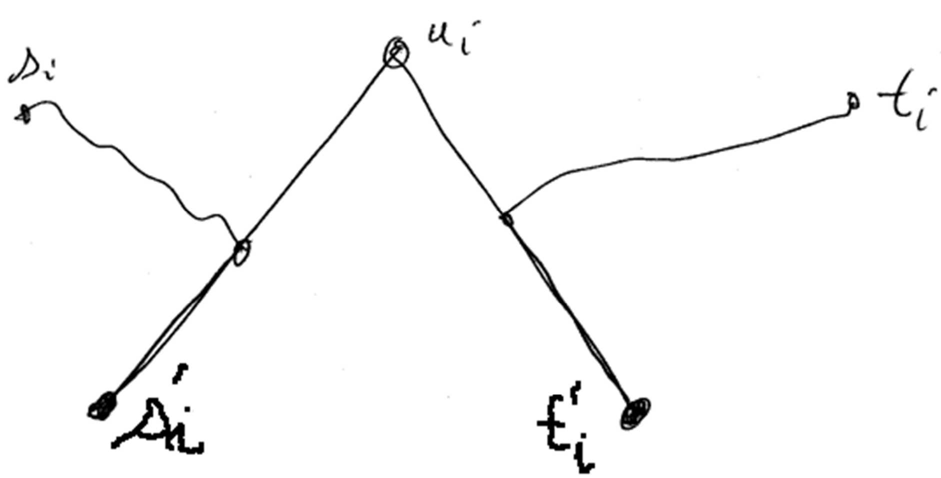
Theorem (Larmani & Mani). There exists a function f such that every $f(k)$-connected graph is k-linked.

Proof. By the previous theorem WMA G has a K_{3k} subdivision.



Let U be the set of branch vertices of the K_{3k} subdivision. Pick a set $\mathcal{P}$ of $2k$ disjoint paths from $\{s_1, \dots, s_k, t_1, \dots, t_k\}$ to U with as few edges outside the subdivision as possible. Note $|U| = 3k$; let $u_1, u_2, \dots, u_k \in U$ be not used by those paths.

Look at the member of $\mathcal{P}$ that connects s_i to say $s_i' \in U$ and the member that connects t_i to say $t_i' \in U$. Then look at the paths in the K_{3k} subdivision between s_i' and u_i, and between t_i' and u_i. Finally, look at the closest vertices to u_i on those paths that belong to $\mathcal{P}$. The minimality of $\mathcal{P}$ implies



Connect s_i to t_i through u_i. □

Application to elementary school mathematics

Find MAX_n, the maximum of n numbers, in k rounds using p processors. In each round each processor can compare two numbers. There is no limit on the amount of communication among the processors.

For $k = 1$ need $\binom{n}{2}$ processors.

Theorem. MAX_n can be solved in two rounds using $O(n^{4/3})$ processors.

Proof. Divide the numbers into $n^{2/3}$ groups, each of size $n^{1/3}$. In each group, find MAX using $O(n^{2/3})$ processors. That's $O(n^{4/3})$ processors total. That was round one. In round two find the max among the $n^{2/3}$ winners using $O(n^{4/3})$ processors. □

Corollary (reminder) If G has n vertices and e edges, then

$$\alpha(G) \geq \frac{n^2}{2e+n}$$

Theorem. Any algorithm that finds MAX_n in two rounds must use $\Omega(n^{4/3})$ processors.

Proof. Let $\mathcal{A}$ be an algorithm that uses p processors. Define G with $V(G) = \{1, \ldots, n\}$ by saying $i \sim j$ if $\mathcal{A}$ compares x_i and x_j in round one. Then G has n vertices and p edges, and so

$$\alpha(G) \geq \frac{n^2}{2p+n}$$

Let I be an independent set in G. In round two we must compare every pair of elements of I, and so

$$p \geq \Omega\left(\left(\frac{n^2}{2p+n}\right)^2\right)$$

If $p \geq n$, then

$$9p^3 = p(2p+p)^2 \geq p(2p+n)^2 \geq \Omega(n^4)$$

and so $p \geq \Omega(n^{4/3})$, as desired.

If $p \leq n$, then

$$p \geq \Omega\left(\frac{n^4}{(2p+n)^2}\right) \geq \Omega\left(\frac{n^4}{(2n+n)^2}\right) = \Omega(n^2)$$

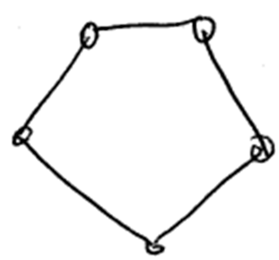
a contradiction. □

Ramsey theory

Observation. Out of 6 people at a party, either some 3 know each other, or some 3 don't know each other.

In other words, if $|V(G)| = 6$, then either $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

Here 6 is best possible:



In general, is it true that for every value of "3" there is a value of "6" such that the above holds?

Definition. Given integers k, ℓ let $r(k, \ell)$ denote the smallest integer N such that every graph G on N vertices has either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Theorem (Special case of Ramsey's theorem). For all integers $k, \ell \geq 2$ the number $r(k, \ell)$ is well-defined and

$$r(k, \ell) \leq r(k, \ell - 1) + r(k - 1, \ell)$$

Examples. $r(k, \ell) = r(\ell, k)$

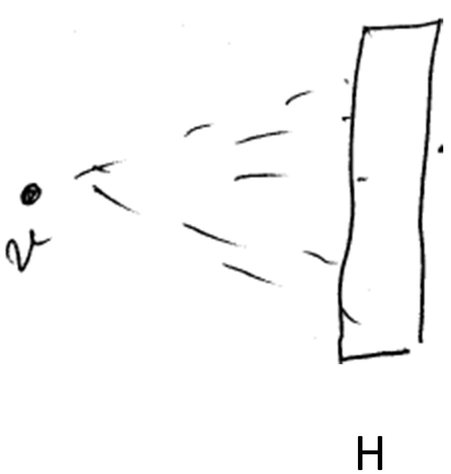
$r(k, 1) = 1$

$r(k, 2) = k$ if $k \geq 2$

Proof. By induction on $k + \ell$. Need to show that every G on $N := r(k, \ell - 1) + r(k - 1, \ell)$ vertices satisfies $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Pick $v \in V(G)$. Either v has $\geq r(k, \ell - 1)$ non-neighbors, or it has $\geq r(k - 1, \ell)$ neighbors.

Case 1. v has $\geq r(k, \ell - 1)$ non-neighbors. Let H be the subgraph of G induced by the non-neighbors of v.



By induction H has a clique of size k or an independent set of size $\ell - 1$. In the latter case add v.

Case 2 is analogous