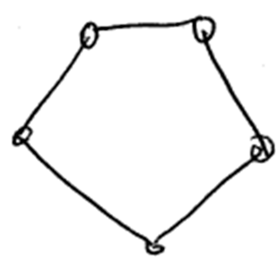
Ramsey theory

Observation. Out of 6 people at a party, either some 3 know each other, or some 3 don't know each other.

In other words, if $|V(G)| = 6$, then either $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

Here 6 is best possible:



In general, is it true that for every value of "3" there is a value of "6" such that the above holds?

Definition. Given integers k, ℓ let $r(k, \ell)$ denote the smallest integer N such that every graph G on N vertices has either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Theorem (Special case of Ramsey's theorem). For all integers $k, \ell \geq 2$ the number $r(k, \ell)$ is well-defined and

$$r(k, \ell) \leq r(k, \ell - 1) + r(k - 1, \ell)$$

Examples. $r(k, \ell) = r(\ell, k)$

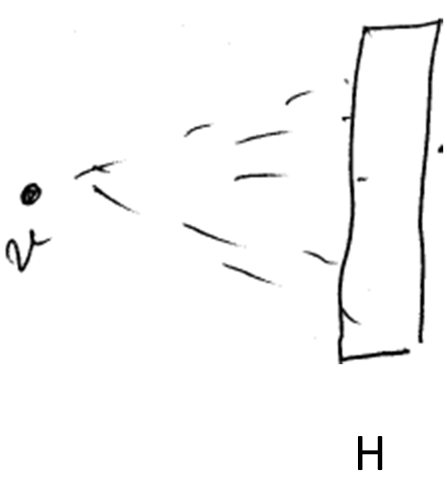
$r(k, 1) = 1$

$r(k, 2) = k$ if $k \geq 2$

Proof. By induction on $k + \ell$. Need to show that every G on $N := r(k, \ell - 1) + r(k - 1, \ell)$ vertices satisfies $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.

Pick $v \in V(G)$. Either v has $\geq r(k, \ell - 1)$ non-neighbors, or it has $\geq r(k - 1, \ell)$ neighbors.

Case 1. v has $\geq r(k, \ell - 1)$ non-neighbors. Let H be the subgraph of G induced by the non-neighbors of v.



By induction H has a clique of size k or an independent set of size $\ell - 1$. In the latter case add v.

Case 2 is analogous

Notation: $[X]^2 :=$ all 2-element subsets of X.

Restatement: For all $k, \ell \geq 2$ there exists an integer N such that for every "coloring" $c: [\{1,2, \dots, N\}]^2 \to \{0,1\}$ there exists either

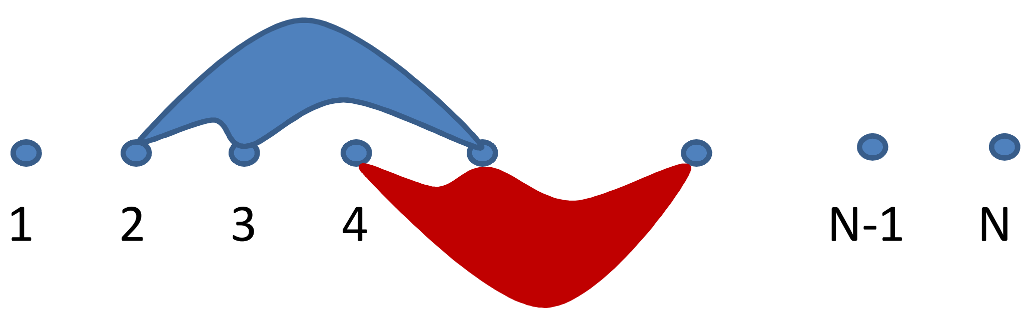
- a set $A \subseteq \{1,2, \dots, N\}$ of size k such that $c(X) = 0 \ \forall \ X \in [A]^2$, or
- a set $B \subseteq \{1,2, \dots, N\}$ of size ℓ such that $c(X) = 1 \ \forall \ X \in [B]^2$.

Definition. For $n \geq 2$ let $r_n(k, \ell)$ be the least integer N such that for every $c: [\{1,2, \dots, N\}]^n \to \{0,1\}$ there exists either

- a set $A \subseteq \{1,2, \dots, N\}$ of size k such that $c(X) = 0 \ \forall \ X \in [A]^n$ or
- a set $B \subseteq \{1,2, \dots, N\}$ of size ℓ such that $c(X) = 1 \ \forall \ X \in [B]^n$

We will say that A is 0-**monochromatic** and that B is 1-**monochromatic**.

Illustration. $n = 3$

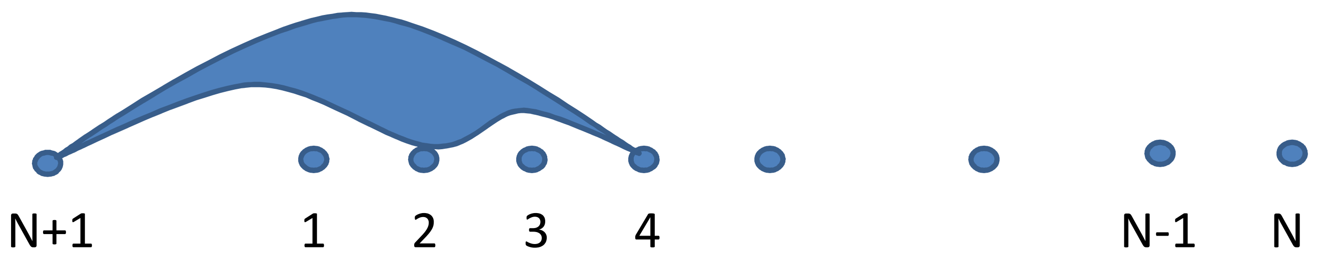


Remark. $r_1(k, \ell) = k + \ell - 1$

Theorem (Ramsey 1930) Let k, ℓ, n be integers with $1 < n < \min\{k, \ell\}$. Then $r_n(k, \ell)$ is well-defined and

$$r_n(k, \ell) \leq r_{n-1}\big(r_n(k-1, \ell), r_n(k, \ell-1)\big) + 1$$

Proof. By induction n, and, subject to that, on $k + \ell$. Let $N \coloneqq r_{n-1}(r_n(k-1, \ell), r_n(k, \ell-1))$. Must show for every coloring c: $[\{1,2, \dots, N, N+1\}]^n \to \{0,1\}$ there exists either a 0-monochromatic set of size k, or a 1-monochromatic set of size ℓ.



Define d: $[\{1,2, \dots, N\}]^{n-1} \to \{0,1\}$ by

$$d(X) = c(X \cup \{N+1\})$$

By induction on n applied to $\{1, \dots, N\}$ and the coloring d there is either

- a set $A \subseteq \{1, \dots, N\}$ of size $r_n(k-1, \ell)$ such that $d(X) = 0$ for every $X \in [A]^{n-1}$ or
- a set $B \subseteq \{1, \dots, N\}$ of size $r_n(k, \ell-1)$ such that $d(X) = 1$ for every $X \in [B]^{n-1}$.

Case 1. Suppose A exists. Apply induction on $k + \ell$ to the coloring c and set A. We get either:

- $C \subseteq A$ of size $k - 1$ such that $c(X) = 0$ for every $X \in [C]^n$, or

- $D \subseteq A$ of size ℓ such that $c(X) = 1$ for every $X \in [D]^n$.

If D exists, then it is as desired.

So WMA C exists. Let $E \coloneqq C \cup \{N + 1\}$. We claim that E is as desired. To see that we need to show that

$$c(Y) = 0 \ \forall \ Y \in [E]^n$$

That is true if $Y \subseteq C$. If $N + 1 \in Y$, then

$$c(Y) = d(Y - \{N + 1\}) = 0$$

and by the definition of d and the choice of A, as desired.

Case 2. The case when B exists is analogous. □

$[A]^n :=$ all n-element subsets of A

$[A]^{<\omega} :=$ all finite subsets of A

Ramsey's theorem (restated) $\forall\, n\ \forall\, k\ \exists\, N\ \forall\, \mathcal{F} \subseteq [\{1, \dots, N\}]^n$
$\exists\, A \subseteq \{1,2, \dots, N\}$ of size k such that either
$[A]^n \subseteq \mathcal{F}$ or $[A]^n \cap \mathcal{F} = \emptyset$.

Infinite Ramsey's theorem $\forall\, n\ \forall\, \mathcal{F} \subseteq [\{1,2, \dots\}]^n$
$\exists$ an infinite set $A \subseteq \{1,2, \dots\}$ such that either
$[A]^n \subseteq \mathcal{F}$ or $[A]^n \cap \mathcal{F} = \emptyset$.

For $n = 2$ this says: Every infinite graph has either an infinite clique or an infinite independent set.

The proof we did can be adapted to prove the infinite version.

Application to number theory

Let $f(S, n)$ denote the number of ways n can be written as $n = a + b$, where $a, b \in S$.

Open problem. Let $f(A, n) \geq 1$ for all n. Does that imply $\limsup_{n\to\infty} f(A, n) = +\infty$?

Multiplicative analogue:

Definition. $X \subseteq \mathbb{N}$ is a *multiplicative base* if for every $n \in \mathbb{N}$ there exist $x, y \in X$ such that $n = xy$.

Theorem (Erdös) If X is a multiplicative base, then for every ℓ there exists an integer $n \in \mathbb{N}$ that can be expressed as a product of two elements of X in at least ℓ ways.

Lemma. Let A be an infinite set, and let $\mathcal{F} \subseteq [A]^{<\omega}$ be such that for every $F \in [A]^{<\omega}$ there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cup F_2 = F$ and $F_1 \cap F_2 = \emptyset$. Then for every ℓ there exists $F \in [A]^{<\omega}$ that can be expressed as above in at least ℓ ways.

Proof of Lemma. Let A and ℓ be given. By Ramsey there exists an infinite set $A_1 \subseteq A$ such that

$$[A_1]^1 \subseteq \mathcal{F} \text{ or } [A_1]^1 \cap \mathcal{F} = \emptyset$$

By another application of Ramsey's theorem there exists an infinite set $A_2 \subseteq A_1$ such that

$$[A_2]^2 \subseteq \mathcal{F} \text{ or } [A_2]^2 \cap \mathcal{F} = \emptyset$$

By repeating this argument for $i = 3,4, \dots \ell - 1$ we finally arrive at an infinite set $A_{\ell-1}$such that

$$[A_{\ell-1}]^{\ell-1} \subseteq \mathcal{F} \text{ or } [A_{\ell-1}]^{\ell-1} \cap \mathcal{F} = \emptyset$$

Let $B \coloneqq A_{\ell-1}$. Thus we have for $i = 1,2, \dots, \ell - 1$

(∗) either $[B]^i \subseteq \mathcal{F}$ or $[B]^i \cap \mathcal{F} = \emptyset$.

Case 1. $[B]^\ell \subseteq \mathcal{F}$. Pick $F \in [B]^{2\ell}$. Then $F = F_1 \cup F_2$, where $F_1 \cap F_2 = \emptyset$, $|F_1| = |F_2| = \ell$ in $\binom{2\ell}{\ell} \geq \ell$ ways and $F_1, F_2 \in [B]^\ell \subseteq \mathcal{F}$.

Case 2. $F \in [B]^\ell - \mathcal{F}$. By hypothesis, $F = F_1 \cup F_2$, where $F_1 \cap F_2 = \emptyset$ and $F_1, F_2 \in \mathcal{F}$. In particular, $F_1, F_2 \neq \emptyset$. Let $j = |F_1|$; then $|F_2| = \ell - j$. Since $|F_1| = j$, $F_1 \in [B]^j \cap \mathcal{F}$ and (∗) implies that $[B]^j \subseteq \mathcal{F}$. Similarly, $[B]^{\ell-j} \subseteq \mathcal{F}$. Thus for every partition $F = F_1 \cup F_2$, $F_1 \cap F_2 = \emptyset$, $|F_1| = j$, $|F_2| = \ell - j$ we have $F_1, F_2 \in \mathcal{F}$. So there are $\binom{\ell}{j} \geq \ell$ ways to express F in the desired way. □

Proof of theorem, assuming lemma: Let A be the set of all primes and let X be a multiplicative base. Define

$$\mathcal{F} \coloneqq \{\{p_1, p_2, \dots, p_k\}: p_1, p_2, \dots, p_k \text{ are distinct primes, } p_1 p_2 \cdots p_k \in X\}$$

To apply the lemma we need to show: $\forall$ finite set F of primes

$\exists F_1, F_2 \in \mathcal{F}$ such that $F_1 \cup F_2 = F$ and $F_1 \cap F_2 = \emptyset$.

Let $n = \prod_{p \in F} p$. Then $\exists\, x, y \in X$ such that $n = xy$.

Let F_1 be the prime divisors of x.

Let F_2 be the prime divisors of y.

Then F_1, F_2 are as desired. Thus the hypothesis of the lemma is satisfied .

Let F be as in Lemma and let $n \coloneqq \prod_{p \in F} p$. Then n is as desired.

$\square$