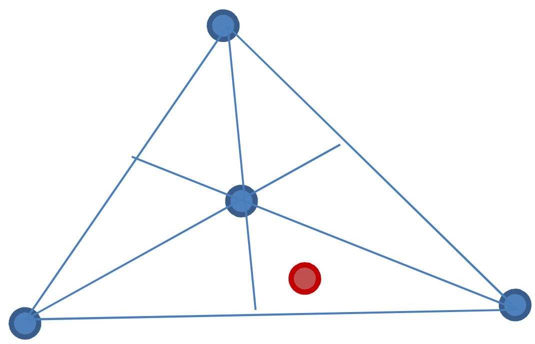
Theorem (Erdös & Szekerés) For every integer n there exists an integer N such that among any N points in the plane in general position (no 3 on a line) there exist n points that form a convex n-gon.

Lemma. Out of 5 points in the plane in general position some four of them form a convex 4-gon.

Proof. Take the convex hull of the 5 points. WMA it is a triangle.



$\square$

Proof of theorem. Let $N \coloneqq r_4(n, 5)$. That is, if we color 4-tuples on $\{1, \dots, N\}$ red and blue, then either there is a set $A \subseteq \{1, \dots, N\}$ of size n such that every 4-tuple in A is red, or there is a set $B \subseteq \{1, \dots, N\}$ of size 5 such that every 4-tuple in B is blue. This N exists by Ramsey's theorem. Let $X \subseteq \mathbb{R}^2$ be a set of N points in general position. Color 4-tuples of X

$$\begin{cases} \text{red} & \text{if they form a convex 4-gon} \\ \text{blue} & \text{otherwise} \end{cases}$$

By the lemma we must get a set A as above.

By Caratheodory's theorem, if some member of A is in the convex hull of the others, then it is in the convex hull of 3 other points ⇒ it and the 3 other points do not form a convex 4-gon, contrary to the 4-tuple being red.

⇒ The set A forms a convex n-gon. □

How big is $r_2(k,\ell)$? We have shown

$$r_2(k,\ell) \leq r_2(k-1,\ell) + r_2(k,\ell-1)$$

Corollary:

$$r_2(k,\ell) \leq \binom{k+\ell-2}{k-1}$$

Proof. Immediate by induction.

Corollary: $r_2(k,k) \leq 4^k$

Theorem (Erdös) $r_2(k,k) \geq 2^{k/2}$ for all $k \geq 2$.

Proof. $r_2(2,2) = 2 \geq 2^{2/2}$. So WMA $k \geq 3$. Let $\mathcal{G}_n$ be the set of all graphs with vertex-set $\{1,2,\dots,n\}$. Then

$$|\mathcal{G}_n| = 2^{\binom{n}{2}}$$

Let $X \subseteq \{1,2,\dots,n\}$ have size k. How many graphs in $\mathcal{G}_n$ have X as a clique: $2^{\binom{n}{2}-\binom{k}{2}}$

How many graphs in $\mathcal{G}_n$ have a clique of size k:

$$\leq \binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}$$

We need to show that for every $n < 2^{k/2}$ there exists a graph on n vertices with no clique or independent set of size k. The proportion of graphs in $\mathcal{G}_n$ that have a clique of size k is

$$\leq \frac{\binom{n}{k} 2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}}} \leq \frac{n^k}{k!} 2^{-\binom{k}{2}} < \frac{\left(2^{\frac{k}{2}}\right)^k 2^{-\binom{k}{2}}}{k!} = \frac{2^{\frac{k^2}{2}-\frac{k^2}{2}+\frac{k}{2}}}{k!} =$$

$$= \frac{2^{k/2}}{k!} = \frac{\left(\sqrt{2}\right)^3}{2 \cdot 3} \cdot \frac{\sqrt{2}}{4} \cdot \frac{\sqrt{2}}{5} \cdot \ldots \cdot \frac{\sqrt{2}}{k} < \frac{1}{2}$$

Thus $< {}^{1}/_{2}$ graphs in $\mathcal{G}_n$ have a clique of size k.

Similarly $< {}^{1}/_{2}$ graphs in $\mathcal{G}_n$ have an independent set of size k.

Therefore there exists a graph that has neither. $\square$

So $$2^{k/2} \leq r_2(k,k) \leq 4^k$$

Does $\lim_{k\to\infty}(r_2\,(k,k))^{1/k}$ exist, and if so, what is it? If it exists, it is between $\sqrt{2}$ and 4.

For $r_n(k,k)$ the bounds are

$$2^{2^{\cdot^{\cdot^{2^{c_1 k}}}}}\Big\}\, n-2 \quad \leq \; r_n(k,k) \leq 2^{2^{\cdot^{\cdot^{2^{c_2 k}}}}}\Big\}\; n-1$$