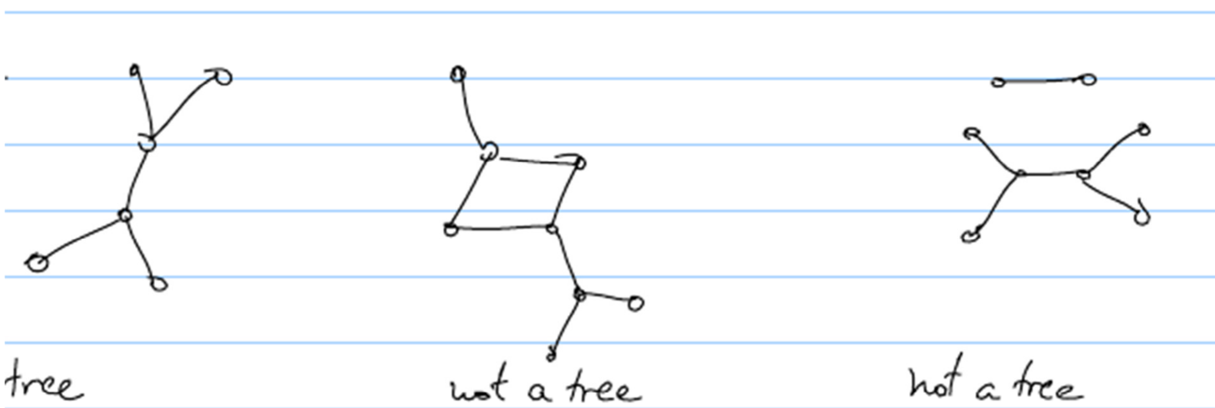
Definition. A ***tree*** is a connected graph with no cycles.

Examples.



Theorem. Let T be a tree. Then

(i) every two vertices of T are connected by exactly one path

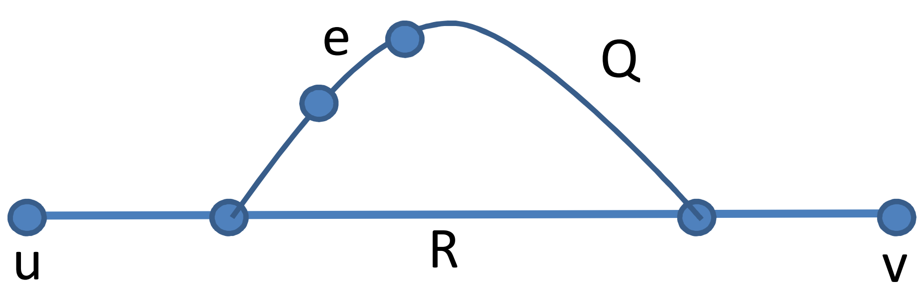
(ii) if $|V(T)| \geq 2$, then T has at least two vertices of degree one

(iii) $|E(T)| = |V(T)| - 1$

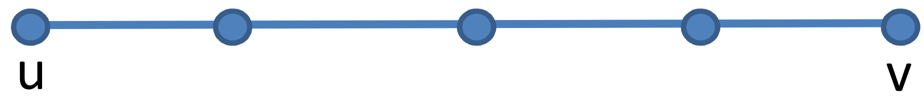
(iv) the deletion of any edge results in exactly two components

(v) if G is the multigraph obtained from T by adding a new edge, then G has exactly one cycle.

Proof. (i) Let $u, v \in V(T)$. There is at least one u-v path because T connected. Suppose P_1, P_2 are two distinct u-v paths in T. Then $\exists e \in E(P_2) - E(P_1)$. The edge e is contained in a subpath Q of P_2 that has both ends in $V(P_1)$ and is otherwise disjoint from P_1. Let R be the subpath of P_1 that joins the ends of Q. Then $Q \cup R$ is a cycle, a contradiction.



(ii) Take a longest path P in T, and let u, v be its ends.



Then u, v do not have a neighbor outside of P by maximality and each has at most one neighbor in P, because T has no cycles. So u, v are two vertices of degree one …

… unless $u = v$.

Since $|V(T)| \geq 2$ and T is connected, P has at least two vertices, and hence $u \neq v$.

(iii) By induction on $|V(T)|$

(iv), (v) exercise.

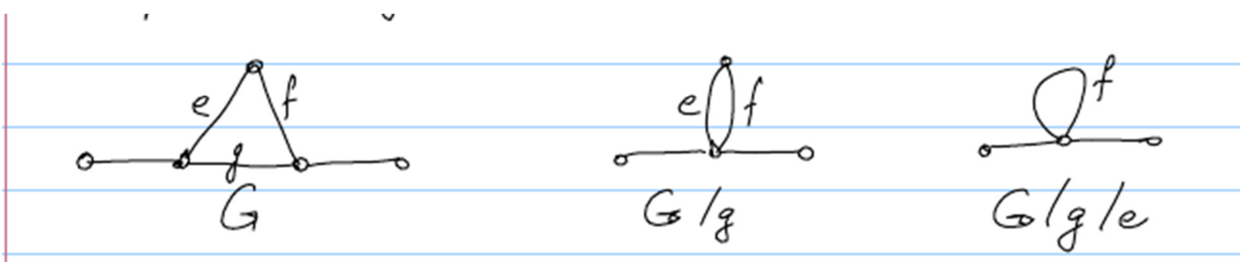
Definition. A ***spanning tree*** of a graph G is a tree T such that T is a subgraph of G and $V(T) = V(G)$.

Note. G has a spanning tree if and only if it is connected.

Note. Knowledge of a minimum weight spanning tree algorithm is assumed.

Let $\tau(G)$ denote the number of spanning trees of G.

G/e is the multigraph obtained by ***contracting*** e; that is, deleting e and identifying its ends. May produce loops or parallel edges:



Proposition. Let G be a multigraph, and let e be an edge that is not a loop. Then

$$\tau(G) = \tau(G\backslash e) + \tau(G/e)$$

Proof. $\tau(G\backslash e) =$ # spanning trees of G that do not use e

$\tau(G/e) =$ # spanning trees of G that use e.□

Note. This gives exponential algorithm (not a good method). We will see a better way.

Definition. Let G be a multigraph with $V(G) = \{1,2,\dots,n\}$ and let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix defined by

$$a_{ij} = \#\text{ edges with ends } i, j$$

Then A is called the ***adjacency matrix*** of G.

The ***Laplacian matrix*** of a graph is defined by $L = (\ell_{ij})$ where

$$\ell_{ij} = \begin{cases} \sum_{k \neq i} a_{ik} & \text{if } i = j \\ -a_{ij} & \text{otherwise} \end{cases}$$

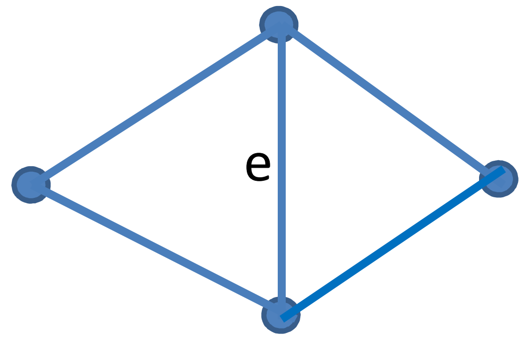
Note that rows and columns sum to 0, and hence $\det L = 0$.

Theorem (Kirchhoff's Matrix Tree Theorem). Let G be a multigraph, let L be its Laplacian matrix, let $k \in \{1,2,\dots,n\}$, and let $L(k)$ denote the matrix obtained from L by deleting the k^{th} row and k^{th} column. Then $\tau(G) = \det L(k)$.

Notice that $\det L(k) = 0$ if G is disconnected.

Example.

$G =$



$\tau(G) = 8$ (4 spanning trees containing e, 4 not containing e).

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

$$L(1) = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$\det L(1) = 3 \times 2 \times 2 - 2 - 2 = 8$

What is $\det(M + E_{vv})$? WMA $v = 1$.

$$\det(M + E_{11}) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (M + E_{11})_{i\sigma(i)} =$$

$$= \sum_{\sigma:\sigma(1)=1} \operatorname{sgn}(\sigma)\,(M_{11} + 1) \prod_{i=2}^{n} M_{i\sigma(i)} +$$

$$+ \sum_{\sigma:\sigma(1)\neq 1} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{i\sigma(i)} =$$

$$= \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} M_{i\sigma(i)} + \sum_{\sigma:\sigma(i)=1} \operatorname{sgn}(\sigma) \prod_{i=2}^{n} M_{i\sigma(i)} =$$

$$= \det(M) + \det M(1)$$

Kirchhoff's Matrix Tree Theorem. Let G be a multigraph, let L be its Laplacian matrix, let $k \in \{1,2,\ldots,n\}$, and let $L(k)$ denote the matrix obtained from L by deleting the k^{th} row and k^{th} column. Then $\tau(G) = \det L(k)$.

Proof. If G is disconnected, then $\tau(G) = 0 = \det L(k)$.

WMA G is connected and loopless.

If $|E(G)| = 0$, then $\tau(G) = 1 = \det L(k)$.

We proceed by induction on $|E(G)|$.

Recall

$$\tau(G) = \tau(G\backslash e) + \tau(G/e)$$

Enough to show

$$(*) \qquad \det L_G(u) = \det L_{G\backslash e}(u) + \det L_{G/e}(w)$$

where $e = uv$ and w is the new vertex of G/e

$$L_G(u) = L_{G\backslash e}(u) + E_{vv}$$

$$\begin{aligned}\det L_G(u) &= \det[L_{G\backslash e}(u) + E_{vv}] \\ &= \det L_{G\backslash e}(u) + \det L_{G\backslash e}(u,v) \\ &= \det L_{G\backslash e}(u) + \det L_{G/e}(w)\end{aligned}$$

This proves $(*)$, and hence the theorem. $\square$

Theorem. (Cayley) $\tau(K_n) = n^{n-2}$. In other words there are exactly n^{n-2} trees with vertex-set $\{1,2,\dots,n\}$.

Proof. By Kirchhoff's Matrix Tree Theorem

$$\tau(K_n) = \det\left.\begin{pmatrix} n-1 & -1 & & -1 \\ -1 & n-1 & & \\ & & & -1 \\ -1 & & -1 & n-1 \end{pmatrix}\right\} n-1 =$$

[by adding rows $2,3,\dots,n-1$ to row 1]

$$= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & n-1 & -1 & -1 \\ \vdots & & & \\ -1 & & -1 & n-1 \end{pmatrix} =$$

[by adding row 1 to all other rows]

$$= \det\begin{pmatrix} 1 & 1 & & 1 \\ & n & & \\ & & & 0 \\ 0 & & & n \end{pmatrix} = n^{n-2}$$

□

Definition. Let G, H be multigraphs. A mapping $f: V(G) \to V(H)$ is an ***isomorphism*** if it is a bijection and for every two vertices $u, v \in V(G)$ the number of edges with ends u, v in G is the same as the number of edges with ends $f(u), f(v)$ in H.

Necessary conditions: $|V(H)| = |V(G)|$, $|E(H)| = |E(G)|$, same degree sequence.