The clique number of a random graph in $\mathcal{G}(n, 1/2)$

Let $X_d(G) \coloneqq \#K_d$-subgraphs in G

$EX_d = \binom{n}{d} 2^{-\binom{d}{2}} =: f(d)$

We will be interested in d s.t. $f(d) \sim 1$. To gain some intuition note

$$f(d) \sim n^d 2^{-d^2/2} = 2^{d \log n - d^2/2}$$

and so $d \sim 2 \log n$. Now let us work rigorously.

$$\frac{f(d)}{f(d+1)} = \frac{\binom{n}{d} 2^{-\binom{d}{2}}}{\binom{n}{d+1} 2^{-\binom{d+1}{2}}} =$$

$$= \frac{n!\,(d+1)!\,(n-d-1)!}{d!\,(n-d)n!} 2^{\frac{(d+1)d}{2} - \frac{d(d-1)}{2}} = \frac{d+1}{n-d} \cdot 2^d$$

This is > 1 if $2^d \geq n$. Thus if $d \geq \log n$, then $f(d)$ is decreasing.

Definition. Choose $d_0 \geq \log n$ such that

$$f(d_0) \geq \log n > f(d_0 + 1)$$

Exercise. $d_0 = 2 \log n + O(\log \log n)$

Theorem. (Bollobás, Erdős, Matula 1976) The clique number of a.e. graph in $\mathcal{G}(n, 1/2)$ is either d_0 or $d_0 + 1$.

½ of Proof.

$$EX_{d_0+2} = f(d_0+2) = \frac{f(d_0+2)}{f(d_0+1)} f(d_0+1) <$$

$$< \frac{n - d_0 - 1}{d_0 + 2} 2^{-d_0-1} \log n \leq$$

$$\leq \frac{n}{2\log n + O(\log\log n)} \cdot 2^{-2\log n + O(\log\log n)} \log n \to 0$$

By Markov's inequality $P[X_{d_0+2} \geq 1] \leq EX_{d_0+2} \to 0$ as $n \to \infty$. Thus a.e. graph in $\mathcal{G}(n, 1/2)$ has clique number $\leq d_0 + 1$.

For the other "half" of the proof we must show

$$P\left[X_{d_0} = 0\right] \to 0 \text{ as } n \to \infty$$

Let $d \coloneqq d_0$.

$$P[X_d = 0] \leq P[|X_d - EX_d| \geq EX_d] \leq \frac{EX_d^2 - E^2X_d}{E^2X_d}$$

$$EX_d^2 = ? \qquad X_d = \sum_{|A|=d} Y_A$$

where $Y_A = \begin{cases} 1 \text{ if } A \text{ is a clique} \\ 0 \quad \text{otherwise} \end{cases}$

$$EX_d^2 = E\left(\sum_{|A|=d} Y_A\right)^2 = E\left(\sum_{(A,B)} Y_A Y_B\right) = \sum_{(A,B)} E(Y_A Y_B)$$

$$= \sum_{\ell=0}^{d} \sum_{\substack{(A,B) \\ |A\cap B|=\ell}} E(Y_A Y_B) = \sum_{\ell=0}^{d} \binom{n}{d}\binom{d}{\ell}\binom{n-d}{d-\ell} 2^{-2\binom{d}{2}+\binom{\ell}{2}}$$

It can be shown that $\frac{EX_d^2 - E^2 X_d}{E^2 X_d} \to 0$, but the calculations are not pleasant. Please refer to Bollobás, Modern Graph Theory.

The chromatic number of graphs in $\mathcal{G}(n, 1/2)$.

The previous theorem gives

$$P[\omega(G) < d_0] \to 0 \text{ as } n \to \infty$$

It can be shown using the so-called generalized Jansen's inequality that

$$(*) \qquad P[\omega(G) < d_0 - 4] \le 2^{-n^{2-\epsilon}}$$

holds for all $\epsilon > 0$ and all sufficiently large n. See [Alon-Spencer, Probabilistic methods].

Theorem (Bollobás) For a.e. graph in $\mathcal{G}(n, 1/2)$.

$$\chi(G) = (1 + o(1)) \frac{n}{2\log_2 n}$$

Proof. We have shown

$$P[\alpha(G) \le d_0 + 1] = P[\omega(G) \le d_0 + 1] \to 1 \text{ as } n \to \infty$$

Since $d_0 = 2 \log n + O(\log \log n)$

$$P[\alpha(G) \le (2 + \epsilon) \log n] \to 1 \text{ as } n \to \infty$$

Therefore

$$\chi(G) \ge \frac{n}{\alpha(G)} \ge \frac{n}{(2 + \epsilon) \log n}$$

with probability approaching 1 as $n \to \infty$.
For the upper bound we show

Claim. Let $m := \left\lfloor \frac{n}{\log^2 n} \right\rfloor$. Then a.e. $G \in \mathcal{G}(n, 1/2)$ satisfies:

$(**)$ every m-element subset of $V(G)$ has an independent set of size $k := d_0 - 4$.

Proof.

$$P[\alpha(G[S]) < d_0 - 4 \text{ for some } m \text{ element set } S \subseteq V(G)] \leq$$

$$\leq \binom{n}{m} P[\alpha(G[S_0]) < d_0 - 4] \leq \binom{n}{m} 2^{-m^{2-\epsilon}} \leq 2^n \cdot 2^{-m^{2-\epsilon}} \leq$$

$$\leq 2^{m^{1+\delta} - m^{2-\epsilon}} \to 0$$

This proves the claim.

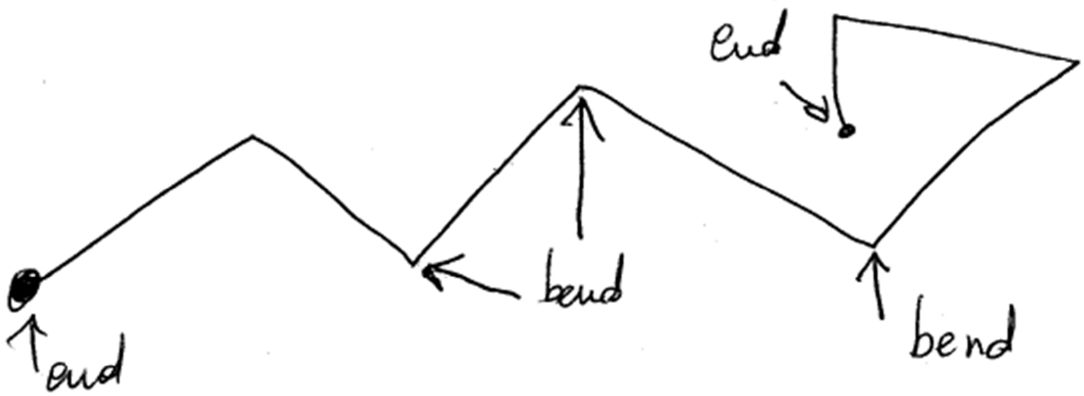
Pick a graph G that satisfies $(**)$. Pull out disjoint independent sets of size $d_0 - 4$ until there is none left. Each such independent set will be a color class. When no independent set of size $d_0 - 4$ is left, then there are $< m$ vertices left. Give each of these vertices different color. Thus

$$\chi(G) \leq \left\lceil \frac{n}{d_0 - 4} \right\rceil + m \leq \frac{n}{(2 - \epsilon') \log n} + o\left(\frac{n}{\log n}\right)$$
$$= (1 + o)) \frac{n}{2 \log n}$$

$\square$

Planar graphs

Definition. A set $A \subseteq \mathbb{R}^2$ homeomorphic to $[0,1]$ that is a union of finitely many straight line segments is called a **polygonal arc.**

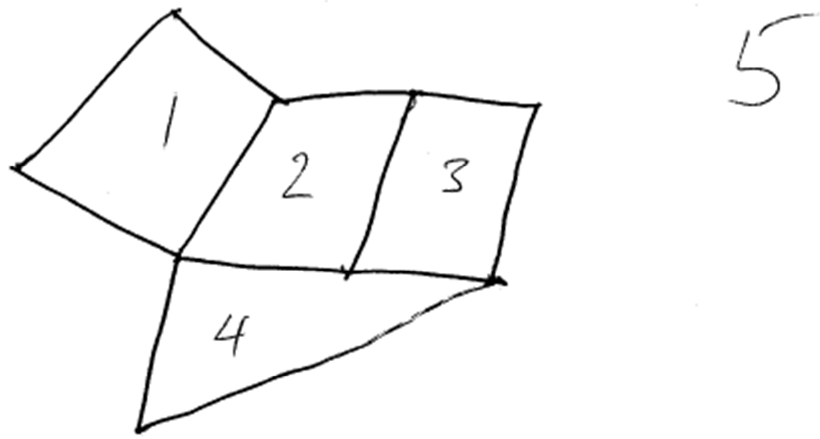


A **polygon** is a set $P \subseteq \mathbb{R}^2$ homeomorphic to

$$\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

that is a union of finitely many straight line segments.

Def. Let $\Omega \subseteq \mathbb{R}^2$ be open. Define $x \sim y$ for $x, y \in \Omega$ to mean that there exists a polygonal arc with ends x, y. This is an equivalence relation. The equivalence classes are called **arcwise connected components** of Ω. If $x \sim y$ for all $x, y \in \Omega$, then Ω is called **arcwise connected**. If $F \subseteq \mathbb{R}$ is closed, then an arcwise connected component of $\mathbb{R}^2 - F$ is called a **face** of F.

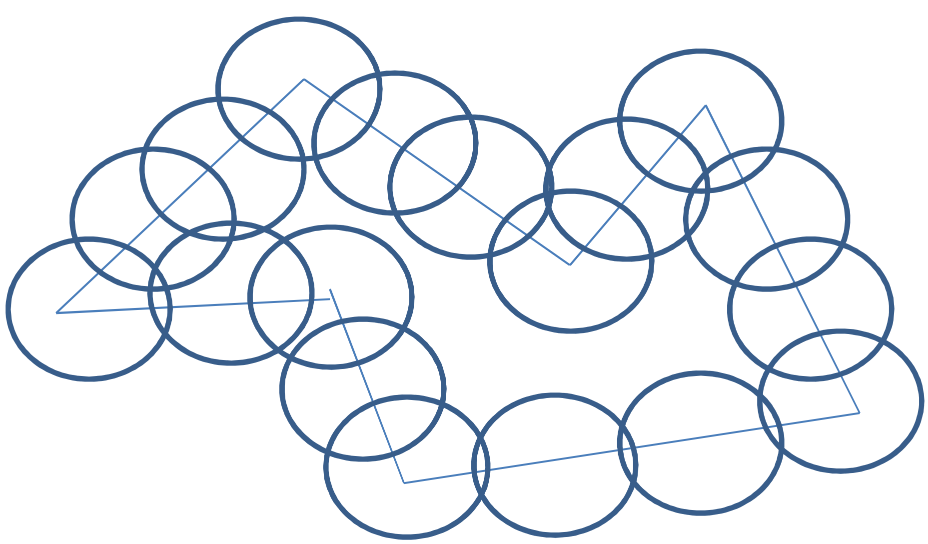


Topological prerequisites:

(1) If $F_1, F_2 \subseteq \mathbb{R}^2$ are closed and at least one of them is bounded, then there exists an $\varepsilon > 0$ such that every point of F_1 is at distance at least ε from every point of F_2.

(2) For every polygon P there exist finitely many open balls $B(x_1), B(x_2), \dots, B(x_k)$ centered at points $x_1, x_2, \dots, x_k \in P$ such that for every $i = 1,2, \dots, k$ the open ball $B(x_i)$ intersects only the segments of P that include x_i and

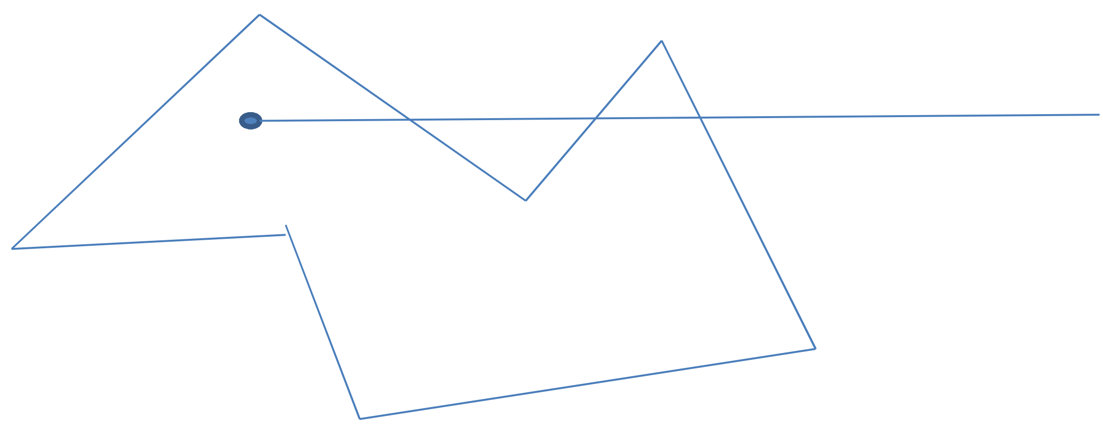
$$P \subseteq B(x_1) \cup B(x_2) \cup \cdots \cup B(x_k).$$



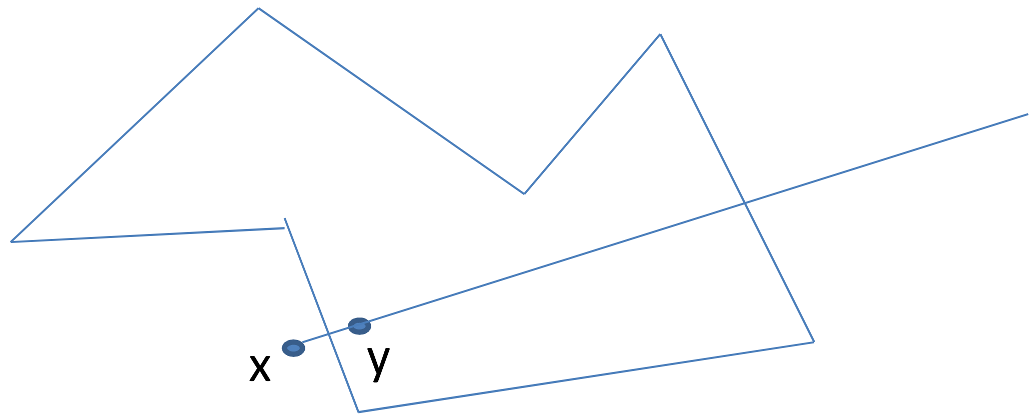
Theorem. (Jordan curve theorem for polygons) Every polygon P has exactly two faces, of which exactly one is bounded. The boundary of each face of P is P.

Proof. Let $x \in \mathbb{R}^2 - P$ and let L be a half-line starting at x. Assume L includes no bend of P. Let

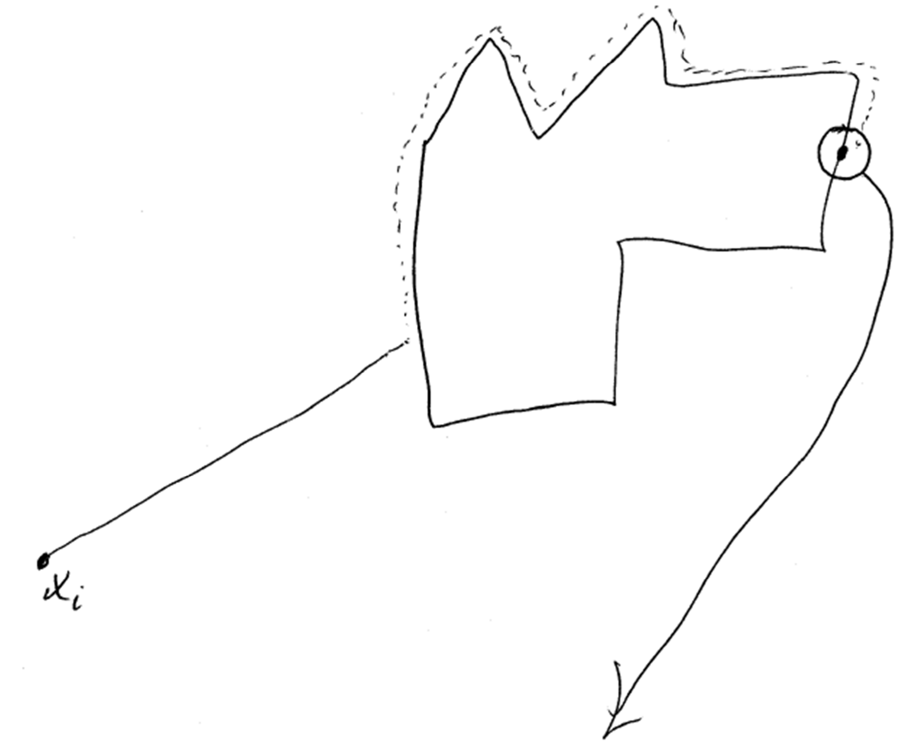
$$\pi(x, L) \coloneqq |P \cap L| \text{ (mod 2)}$$

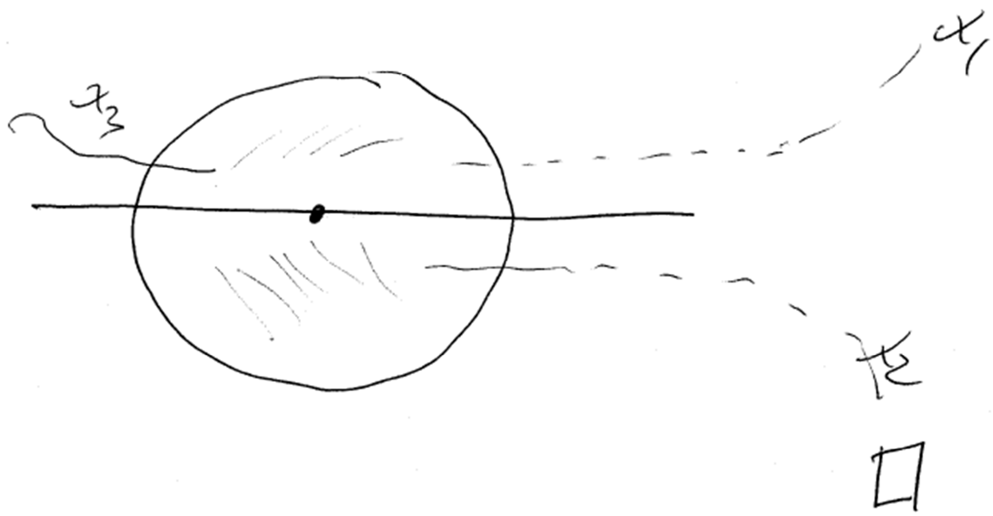


$\pi(x, L)$ can be defined for all L so that it does not depend on L. Call the common value $\pi(x)$. So we have defined a function $\pi: \mathbb{R}^2 - P \to \{0,1\}$. This function is continuous and therefore it is constant on every arcwise connected component of $\mathbb{R}^2 - P$. By considering two points on opposite sides of a segment of P we find that π is onto. Thus P has at least two faces.


x
y

To show there are ≤ 2 faces suppose that x_1, x_2, x_3 belong to different faces. For $i = 1,2,3$ do the following:





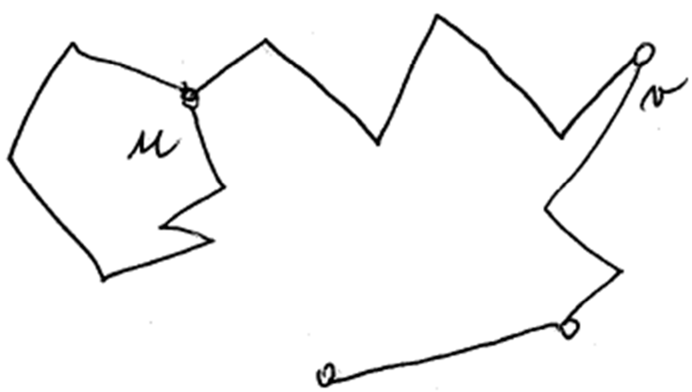
Def. A **plane multigraph** is a multigraph G such that

(i) $V(G) \subseteq \mathbb{R}^2$

(ii) For every non-loop edge $e \in E(G)$ with ends u, v there exists a polygonal arc A with ends u, v such that $e = A - \{u, v\} \subseteq \mathbb{R}^2 - V(G)$.

(iii) for every loop e incident with $u \in V(G)$ there exists a polygon P containing u such that $e = P - \{u\} \subseteq \mathbb{R}^2 - V(G)$, and

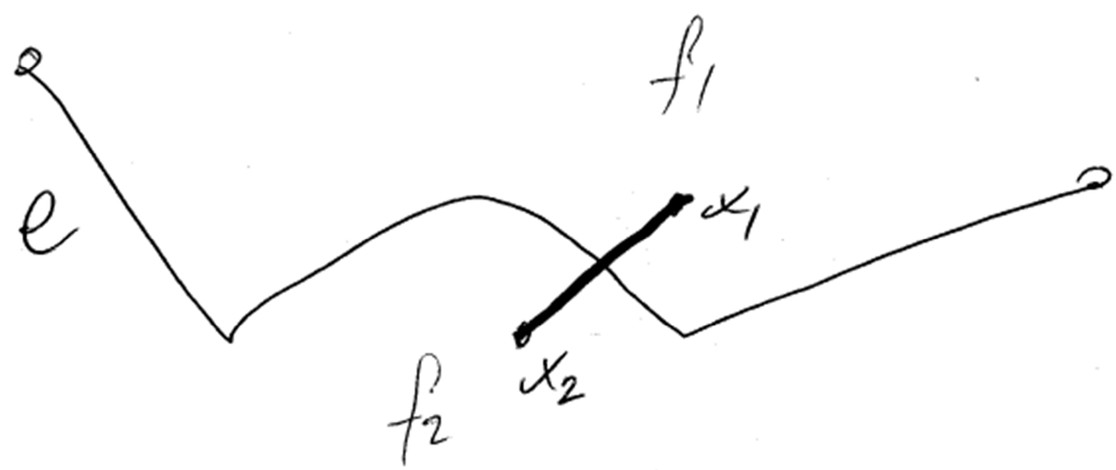
(iv) if $e, e' \in E(G)$ are distinct, then $e \cap e' = \emptyset$.



Def. A graph is **planar** if it is isomorphic to a plane graph Γ. We say Γ is a planar drawing of G. For a plane multigraph G, the **point set** of G is $\bigcup_{e \in E(G)} e \cup V(G)$. We will denote it by G. The set of faces of G is denoted by $F(G)$.

Lemma 1.6. Let G be a plane multigraph, let $e \in E(G)$, let $x_1, x_2 \in \mathbb{R}^2 - G$ be two points such that the straight-line segment x_1x_2 intersects e exactly once and is otherwise disjoint from G. Let f_i be the face of G containing x_i. Then e is a subset of the boundary of both f_1 and f_2, and is disjoint from the boundary of every other face of G. Furthermore, if G is a cycle, then $f_1 \neq f_2$.

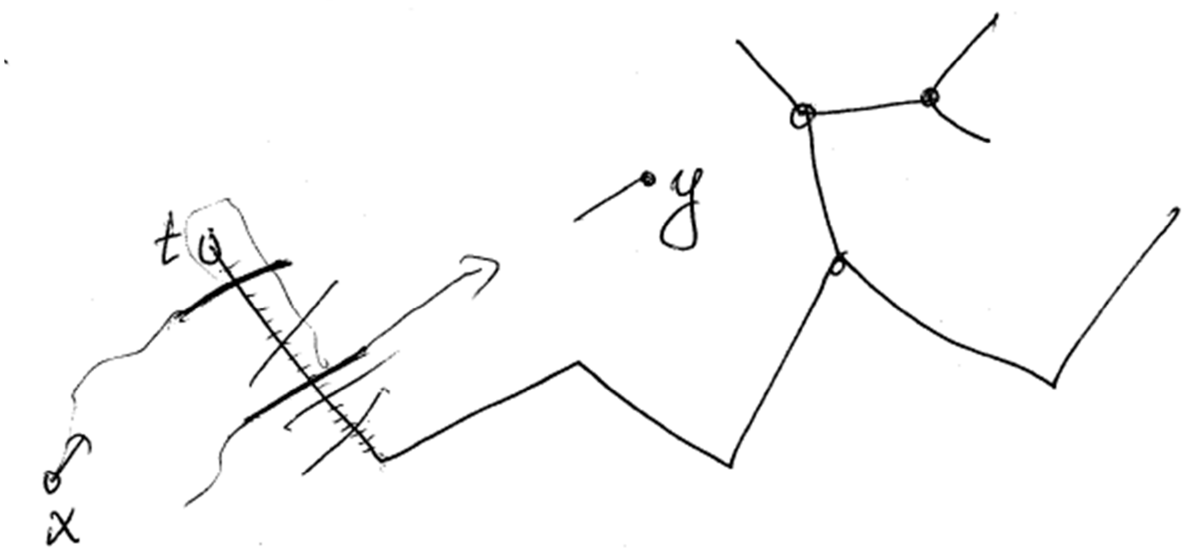
Proof. Similar to the Jordan curve theorem.



Corollary. The boundary of a face of G is the point set of a subgraph of G.

Def. If e, f_1, f_2 are as in the lemma, then we say that f_1, f_2 are the two **faces incident with** e. Possibly $f_1 = f_2$.

Lemma. Let G be a plane multigraph that is a forest. Then $|F(G)| = 1$.



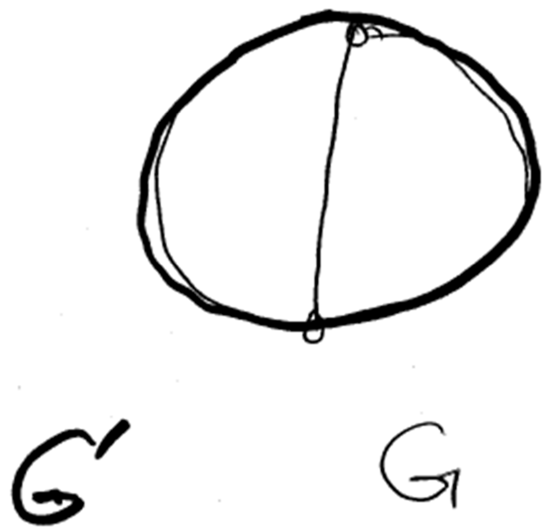
Proof. Induction on the number of bends. □

Lemma 1.10. Let G' be a subgraph of a plane multigraph G. Then

(i) every face of G is a subset of a face of G'

(ii) if $f \in F(G)$ and $bd(f) \subseteq G'$, then $f \in F(G')$

(iii) if $f' \in F(G')$ is disjoint from G, then $f' \in F(G)$.



Proof. (i) easy

(ii) Let $f \in F(G)$. By (i) $\exists\, f' \in F(G')$, $f \subseteq f'$. WMA $f \subsetneqq f'$, for o.w. $f = f' \in F(G')$. So $\exists\, x' \in f' - f$. Pick $x \in f$ and a polygonal arc $A \subseteq f'$ with ends x, x'. Since $A \nsubseteq f$ $\exists\, z \in A \cap bd(f)$. But $z \in A \subseteq f'$, and hence $z \notin G'$. So $bd(f) \nsubseteq G'$, a contradiction.

(iii) Let $f' \in F(G')$ be disjoint from G. Thus f' is an arcwise-connected subset of $\mathbb{R}^2 - G$, and hence $f' \subseteq f$ for some $f \in F(G)$. By (i) $f \subseteq f''$ for some $f'' \in F(G')$. But $f' \subseteq f \subseteq f''$ and so $f' = f''$, because both are faces of G'. Now $f' \subseteq f \subseteq f'' = f'$, and so $f' = f \in F(G)$, as desired. □