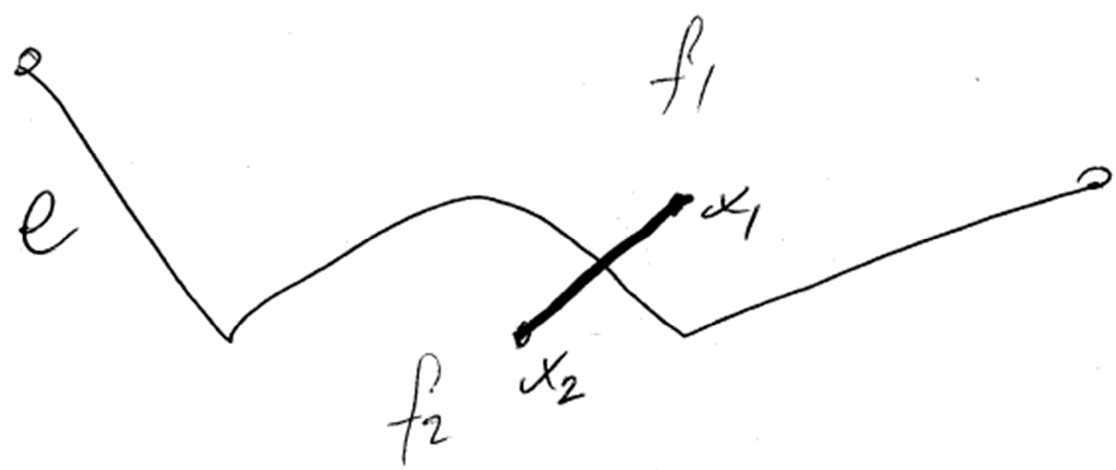
**Lemma 1.6.** Let $G$ be a plane multigraph, let $e \in E(G)$, let $x_1, x_2 \in \mathbb{R}^2 - G$ be two points such that the straight-line segment $x_1x_2$ intersects $e$ exactly once and is otherwise disjoint from $G$. Let $f_i$ be the face of $G$ containing $x_i$. Then $e$ is a subset of the boundary of both $f_1$ and $f_2$, and is disjoint from the boundary of every other face of $G$. Furthermore, if $G$ is a cycle, then $f_1 \neq f_2$.

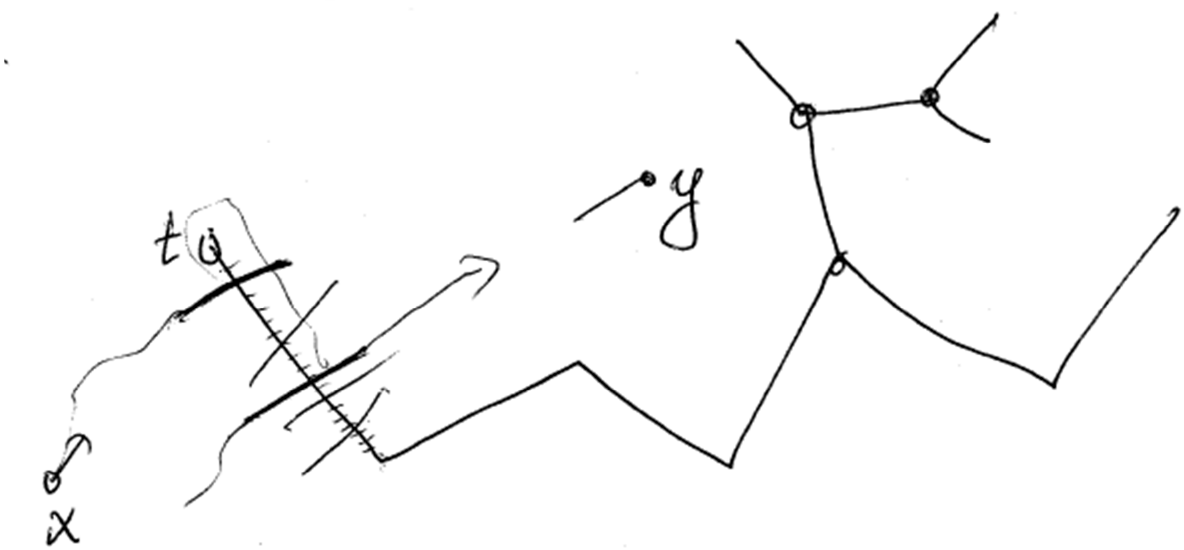
**Proof.** Similar to the Jordan curve theorem.




**Corollary.** The boundary of a face of $G$ is the point set of a subgraph of $G$.

**Def.** If $e, f_1, f_2$ are as in the lemma, then we say that $f_1, f_2$ are the two **faces incident with** $e$. Possibly $f_1 = f_2$.

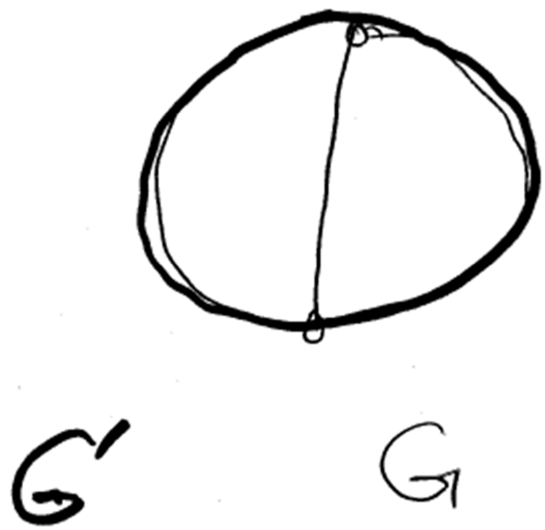
**Lemma.** Let $G$ be a plane multigraph that is a forest. Then $|F(G)| = 1$.




**Proof.** Induction on the number of bends. $\square$

**Lemma 1.10.** Let $G'$ be a subgraph of a plane multigraph $G$. Then

(i) every face of $G$ is a subset of a face of $G'$
(ii) if $f \in F(G)$ and $bd(f) \subseteq G'$, then $f \in F(G')$
(iii) if $f' \in F(G')$ is disjoint from $G$, then $f' \in F(G)$.

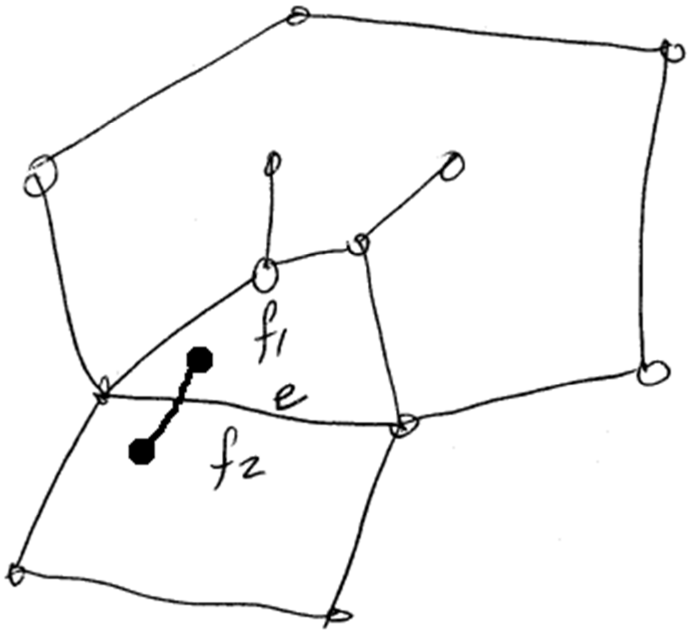



**Proof.** (i) easy

(ii) Let $f \in F(G)$. By (i) $\exists\, f' \in F(G')$, $f \subseteq f'$. WMA $f \subsetneqq f'$, for o.w. $f = f' \in F(G')$. So $\exists\, x' \in f' - f$. Pick $x \in f$ and a polygonal arc $A \subseteq f'$ with ends $x, x'$. Since $A \nsubseteq f$ $\exists\, z \in A \cap bd(f)$. But $z \in A \subseteq f'$, and hence $z \notin G'$. So $bd(f) \nsubseteq G'$, a contradiction.

(iii) Let $f' \in F(G')$ be disjoint from $G$. Thus $f'$ is an arcwise-connected subset of $\mathbb{R}^2 - G$, and hence $f' \subseteq f$ for some $f \in F(G)$. By (i) $f \subseteq f''$ for some $f'' \in F(G')$. But $f' \subseteq f \subseteq f''$ and so $f' = f''$, because both are faces of $G'$. Now $f' \subseteq f \subseteq f'' = f'$, and so $f' = f \in F(G)$, as desired.

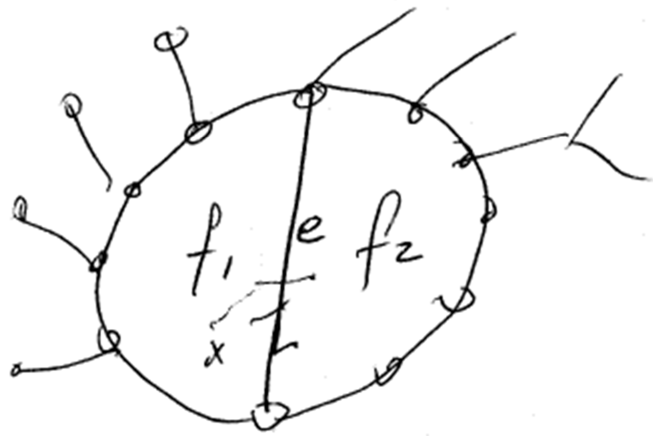
**Lemma 1.11.** Let $e$ be an edge of a plane multigraph $G$ such that $e$ belongs to a cycle of $G$. Let $f_1, f_2$ be the two faces incident with $e$. Then $f_1 \neq f_2$.




**Proof.** Let $C$ be a cycle in $G$ containing $e$. For $i = 1{,}2$ let $f_i \subseteq f_i'$, where $f_i' \in F(C)$. Pick points $x_i \in f_i$ as in Lemma 1.6; that is, the straight-line segment $x_1 x_2$ intersects $e$ exactly once and has no other intersections with $G$. Then $x_i \in f_i'$ and by Lemma 1.6 $f_1' \neq f_2'$. It follows that $f_1 \neq f_2$.

□

**Lemma 1.12.** Let $G$ be a plane graph, let $e \in E(G)$, and let $f_1, f_2$ be the two faces incident with $e$. Let $f_{12} := f_1 \cup e \cup f_2$. Then $f_{12} \in F(G \backslash e)$ and $F(G) - \{f_1, f_2\} = F(G \backslash e) - \{f_{12}\}$.




**Proof.** Note $f_{12}$ is arcwise connected. Thus $f_{12} \subseteq f'$, where $f' \in F(G \backslash e)$. WMA $f_{12} \subsetneqq f'$, for otherwise we are done. Thus there exists a polygonal arc $A$ with $x \in f_{12}$ and $y \in f' - f_{12}$ such that $A \subseteq f'$. By considering a subset of $A$ WMA that $A \cap e = \emptyset$. Since $y \in f' - f_{12} \subseteq \mathbb{R}^2 - G - f_1 - f_2$ it follows that $y$ belongs to a face of $G$ other than $f_1$ or $f_2$, and yet $A$ connects $x$ to $y$ and is disjoint from $G$, a contradiction. This proves $f_{12} \in F(G \backslash e)$.

Let $f \in F(G) - \{f_1, f_2\}$. By Lemma 1.6 $bd(f)$ is disjoint from $e$, and so $bd(f) \subseteq G \backslash e$. Thus $f \in F(G \backslash e)$ by Lemma 1.10 (ii).

Let $f' \in F(G \backslash e) - \{f_{12}\}$. Then $f' \cap G = \emptyset$ and hence $f' \in F(G)$ by Lemma 1.10 (iii). Also, $f' \notin \{f_1, f_2\}$, because $f_1, f_2 \notin F(G \backslash e)$.

$\square$

**Theorem.** (Euler's formula) Every **connected plane** graph $G$ satisfies

$$|V(G)| + |F(G)| = |E(G)| + 2$$

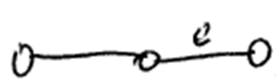
**Proof.** Induction on $|E(G)|$. If $G$ has no cycles, then it is a tree and $|V(G)| = |E(G)| + 1$ and $|F(G)| = 1$. So WMA what $G$ has a cycle $C$. Let $e \in E(C)$, and let $f_1, f_2$ be the incident faces. Then $f_1 \neq f_2$ by Lemma 1.11. By induction applied to $G \backslash e$ we deduce

$$|V(G\backslash e)| + |F(G\backslash e)| = |E(G\backslash e)| + 2$$

But $|V(G)| = |V(G\backslash e)|$, $|E(G)| = |E(G\backslash e)| + 1$ and $|F(G)| = |F(G\backslash e)| + 1$ by Lemma 1.12. □

**Corollary 1.14.** Every **simple** planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges. Moreover, if $G$ has no triangles, then it has at most $2n - 4$ edges.

**Proof.** WMA $G$ is connected. Let $q$ be the number of edge-face incidences $(e, f)$, with the proviso that if $e$ is incident with $f_1$ and $f_2$, and $f_1 = f_2$, then the incidence $(e, f_1) = (e, f_2)$ is counted twice. Then $q = 2|E(G)|$. Since $G$ has no loops or parallel edges, each face contributes at least 3 toward $q$.




Thus $q \geq 3|F(G)|$. So $|F(G)| \leq \frac{2}{3}|E(G)|$. Substituting into Euler's formula

$$|V(G)| + \frac{2}{3}|E(G)| \geq |V(G)| + |F(G)| = |E(G)| + 2$$

$$\frac{1}{3}|E(G)| \leq |V(G)| - 2$$

$$|E(G)| \leq 3|V(G)| - 6.$$

□

**Corollary 1.15.** $K_5$ and $K_{3,3}$ are not planar.

**Proof.**

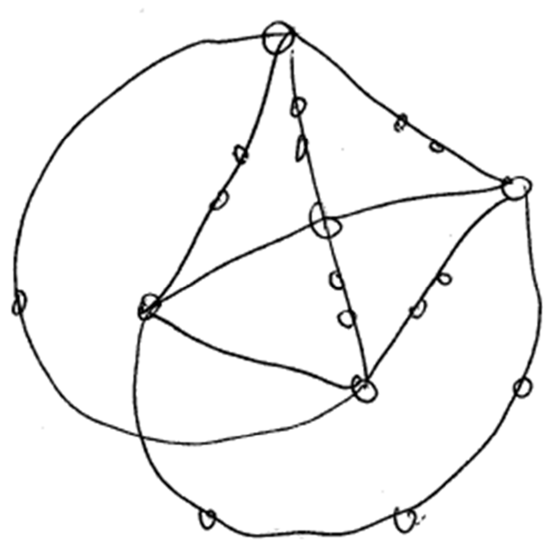
$$|E(K_5)| = 10 \nleq 9 = 3|V(K_5)| - 6$$

$$\left|E\left(K_{3,3}\right)\right| = 9 \leq 12 = 3\left|V\left(K_{3,3}\right)\right| - 6$$

$$\left|E\left(K_{3,3}\right)\right| = 9 \nleq 8 = 2\left|V\left(K_{3,3}\right)\right| - 4$$

□

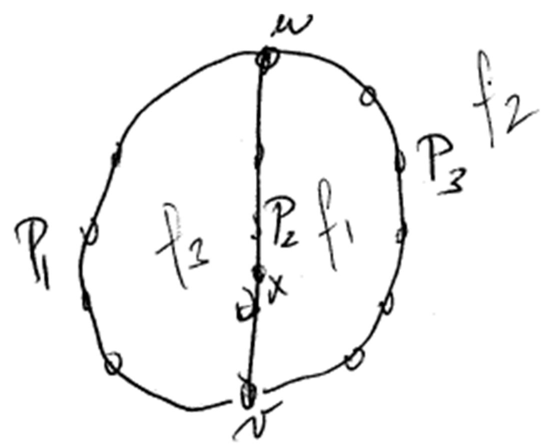
**Reminder .** A subdivision of a graph



No subdivision of $K_5$ or $K_{3,3}$ is planar. No graph that has a $K_5$ or $K_{3,3}$ subdivision is planar. Are there other nonplanar graphs?

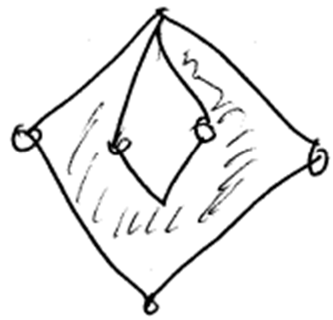
No. That is what Kuratowski's theorem tells us.

**Lemma.** Let $G$ be a plane graph consisting of two vertices and three internally disjoint paths $P_1, P_2, P_3$ joining them. Then $G$ has precisely three faces with boundaries $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$, respectively.



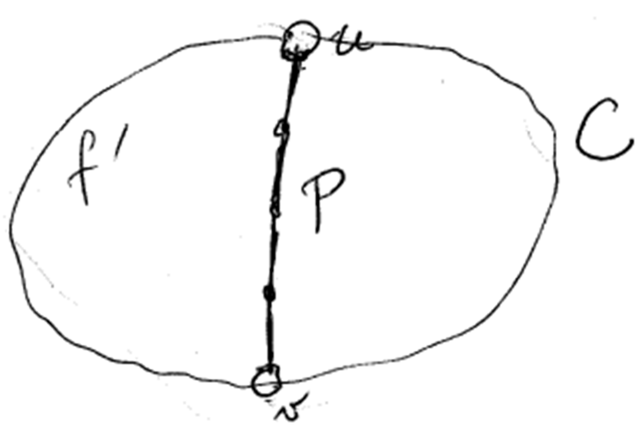

**Proof.** The graph $P_1 \cup P_2$ has exactly two faces by the Jordan curve theorem; let $f_3$ be the one disjoint from $P_3$. Define $f_1, f_2$ similarly. Then $f_1, f_2, f_3 \in F(G)$ by Lemma 1.10 (iii), and they are distinct, because they have different boundaries. Let $f \in F(G)$ and let $x \in bd(f) - V(G)$. WMA $x \in P_2$. But $P_2$ is only incident with $f_1$ and $f_3$, and so $f = f_1$ or $f = f_3$ as desired. □

**Theorem.** Let $G$ be a 2-connected plane graph. Then every face of $G$ is bounded by a cycle.



**Proof.** By induction on $|E(G)|$. If $G$ is a cycle, then done by the Jordan curve theorem. By the ear-decomposition theorem $G$ can be written as $G = G' \cup P$, where $P$ is a path with both ends in $G'$ and otherwise disjoint from it (and $|E(P)| \geq 1$). By induction every face of $G'$ is bounded by a cycle.

Let $f \in F(G)$. Then $f \subseteq f'$, where $f' \in F(G')$. Then $f'$ is bounded by a cycle $C$ of $G'$. Let $P^o$ be the point set $P - \{u, v\}$, where $u, v$ are the ends of $P$.



Since $P^o$ is arcwise-connected, it is a subset of a face of $G'$.

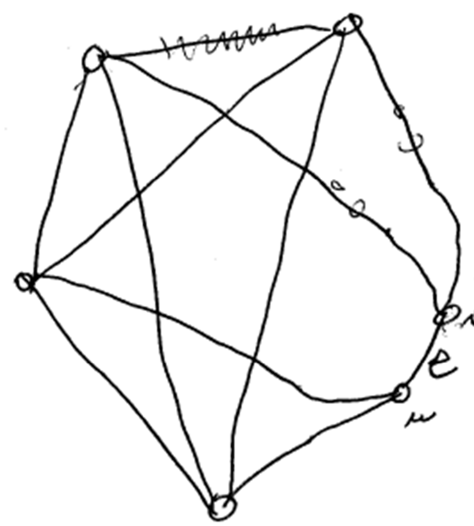
**Case 1.** $P^o \cap f' = \emptyset$. Then $bd(f')$ is disjoint from $G$, and hence $f' \in F(G)$ by Lemma 1.10. Thus $f = f'$ and the boundary of $f = f'$ is $C$, as desired.

**Case 2.** $P^o \subseteq f'$. Now $bd(f) \subseteq \overline{f'} \cap G \subseteq C \cup P \Rightarrow f$ is a face of $C \cup P$ by Lemma 1.10 $\Rightarrow f$ is bounded by a cycle by Lemma 2.2.

$\square$

**Def.** A graph $H$ is a **minor** of $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. An ***H* minor** is a minor isomorphic to $H$.

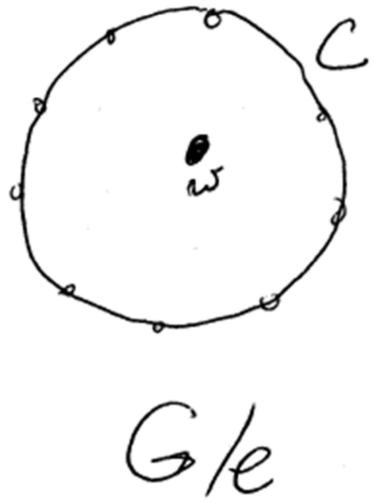
$H$ subdivision $\Rightarrow$ $H$ minor

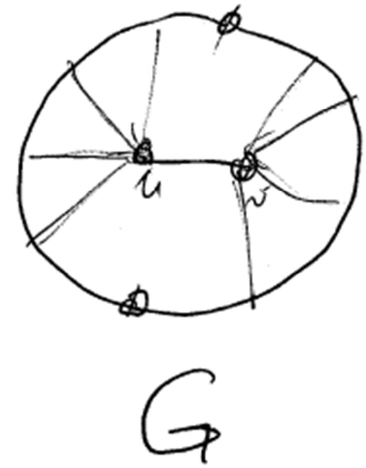



**Theorem** (Special case of Kuratowski's theorem) Let $G$ be a 3-connected graph with no minor isomorphic to $K_5$ or $K_{3,3}$. Then $G$ is planar.

**Proof.** By induction on $|V(G)|$. If $|V(G)| = 4$, then clear. So WMA $|V(G)| \geq 5$. By an old lemma $\exists\ e \in E(G)$ such that $G/e$ is 3-connected. Since $G/e$ has no minor isomorphic to $K_5$ or $K_{3,3}$, it is planar by the induction hypothesis.

Let $e = uv$, let $w$ be the new vertex of $G/e$.

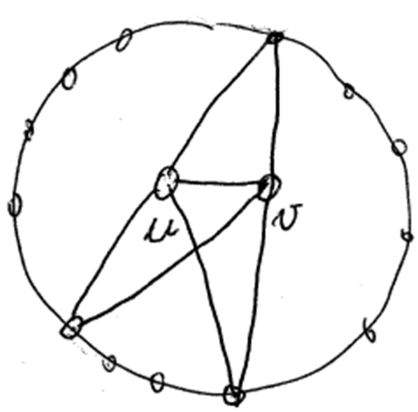



Note $G/e\backslash w$ is 2-connected, and so the face containing $w$ is bounded by a cycle, say $C$.

**Claim.** $C$ can be written as $P_1 \cup P_2$, where $P_1, P_2$ are edge-disjoint paths such that $u$ has all neighbors in $V(P_1) \cup \{v\}$ and $v$ has all neighbors in $V(P_2) \cup \{u\}$.

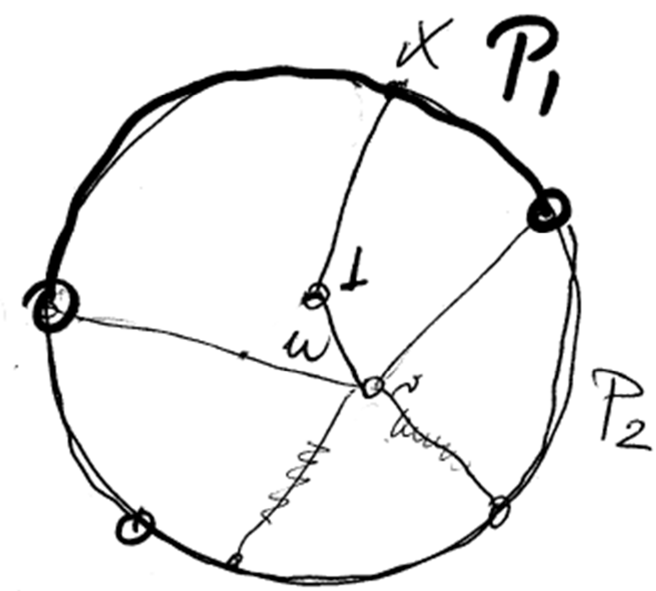
**Proof of claim. Case 1.** Every neighbor of $u$ on $C$ is a neighbor of $v$ and vice versa.



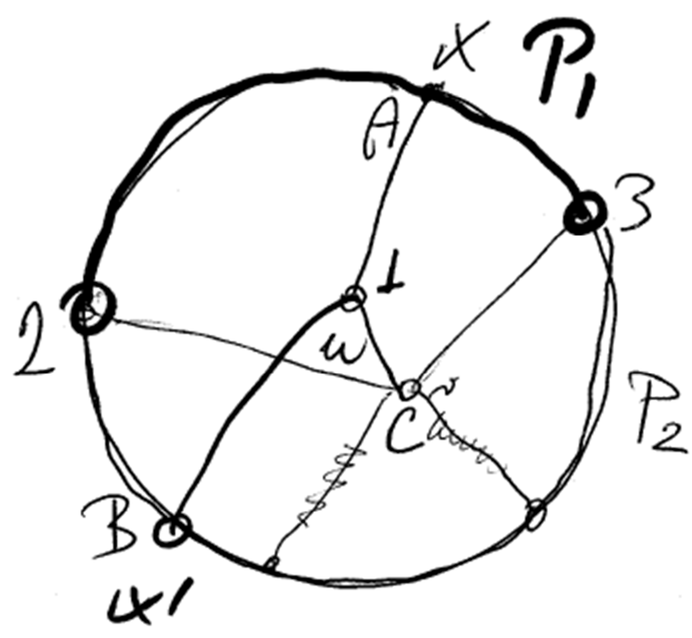

$K_5$ subdivision $\Rightarrow$ $K_5$ minor

**Case 2.** WMA $u$ has a neighbor $x$ on $C$ that is not a neighbor of $v$.

Let $P_1$ be the shortest subpath of $C$ whose ends are neighbors of $v$.




If $u$ has all neighbors in $C$ on $P_1$ $\Rightarrow$ claim holds, so WMA not.
Then $G$ has a $K_{3,3}$ subdivision.




□