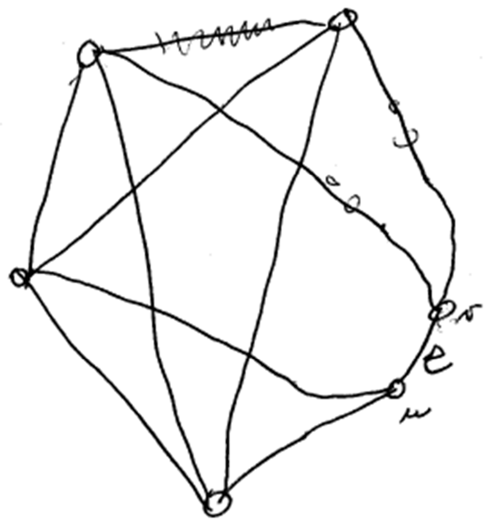
Def. A graph H is a **minor** of G if H can be obtained from a subgraph of G by contracting edges. An ***H* minor** is a minor isomorphic to H.

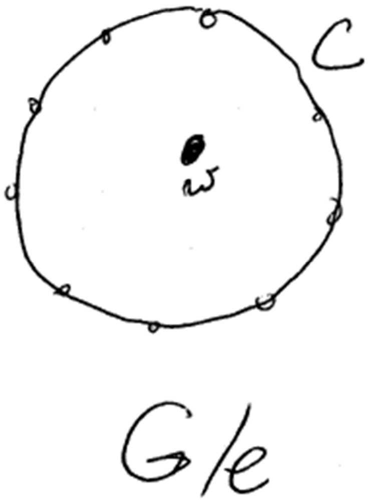
H subdivision $\Rightarrow$ H minor

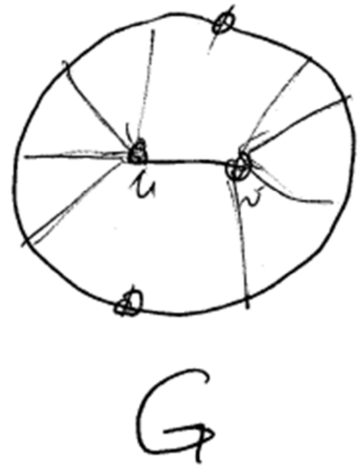


Theorem (Special case of Kuratowski's theorem) Let G be a 3-connected graph with no minor isomorphic to K_5 or $K_{3,3}$. Then G is planar.

Proof. By induction on $|V(G)|$. If $|V(G)| = 4$, then clear. So WMA $|V(G)| \geq 5$. By an old lemma $\exists\ e \in E(G)$ such that G/e is 3-connected. Since G/e has no minor isomorphic to K_5 or $K_{3,3}$, it is planar by the induction hypothesis.

Let $e = uv$, let w be the new vertex of G/e.

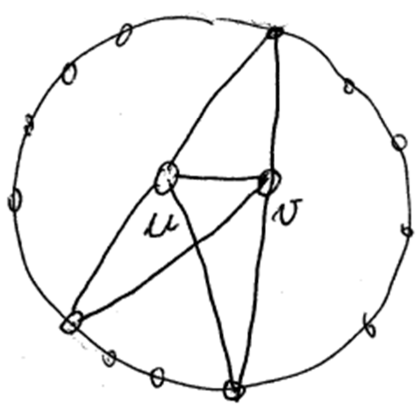




Note $G/e \backslash w$ is 2-connected, and so the face containing w is bounded by a cycle, say C.

Claim. C can be written as $P_1 \cup P_2$, where P_1, P_2 are edge-disjoint paths such that u has all neighbors in $V(P_1) \cup \{v\}$ and v has all neighbors in $V(P_2) \cup \{u\}$.

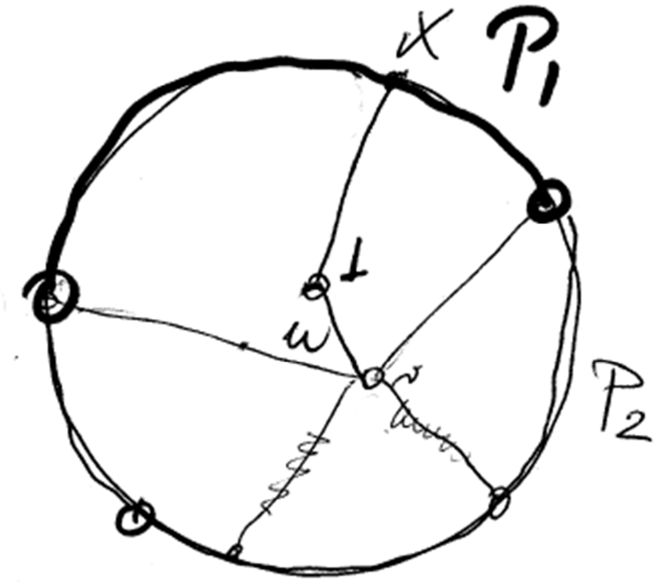
Proof of claim. Case 1. Every neighbor of u on C is a neighbor of v and vice versa.



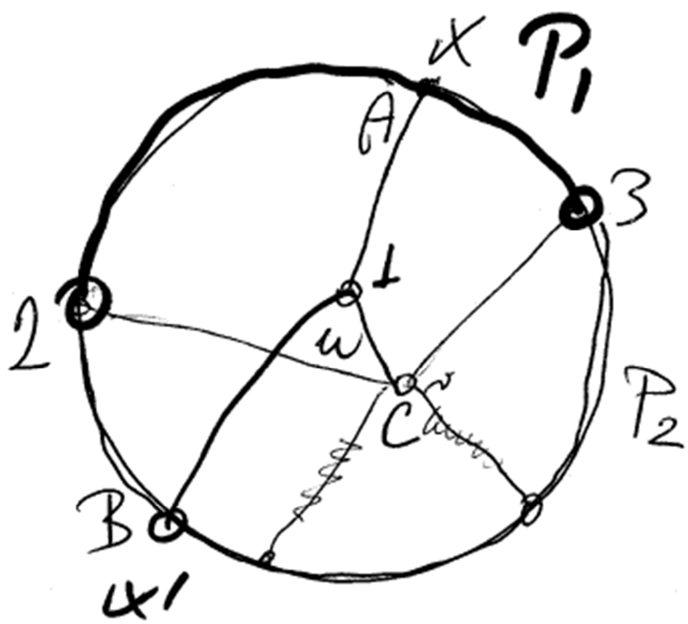
K_5 subdivision $\Rightarrow$ K_5 minor

Case 2. WMA u has a neighbor x on C that is not a neighbor of v.

Let P_1 be the shortest subpath of C whose ends are neighbors of v.



If u has all neighbors in C on P_1 $\Rightarrow$ claim holds, so WMA not.
Then G has a $K_{3,3}$ subdivision.



□

Def. We say that a plane graph G is a **convex drawing** if every edge $e \in E(G)$ is a straight-line segment and every face is bounded by a convex polygon. The previous proof shows:

Theorem. Every 3-connected simple plane graph has a convex drawing.

Lemma 2.12. A graph G has a K_5 or $K_{3,3}$ minor if and only if it has a K_5 or $K_{3,3}$ subdivision.

Proof. $\Leftarrow$: always true.

$\Rightarrow$: $K_{3,3}$ minor implies $K_{3,3}$ subdivision.

K_5 minor implies K_5 or $K_{3,3}$ subdivision (exercise) □

Lemma 2.13. Let G be a graph on ≥ 4 vertices with no K_5 or $K_{3,3}$ subdivision, and assume that adding an edge joining any pair of non-adjacent vertices creates a K_5 or $K_{3,3}$ subdivision. Then G is 3-connected.

Theorem. For a graph G TFAE:

(1) G is planar
(2) G has a straight-line drawing
(3) G has no K_5 or $K_{3,3}$ minor
(4) G has no K_5 or $K_{3,3}$ subdivision.

Remarks. (1) $\Leftrightarrow$ (4) is "Kuratowski's theorem" 1930

(1) $\Leftrightarrow$ (2) is called Fáry's theorem.

Proof. (3) ⇔ (4) by Lemma 2.12.

(2) ⇒ (1) trivial

(1) ⇒ (4) by Corollary 2.1.

To prove (3) ⇒ (2) let G have no K_5 or $K_{3,3}$ minor. WMA G is edge-maximal. Then G is 3-connected by Lemma 2.13. ⇒ G has a straight-line drawing by the Special case of Kuratowski's theorem.

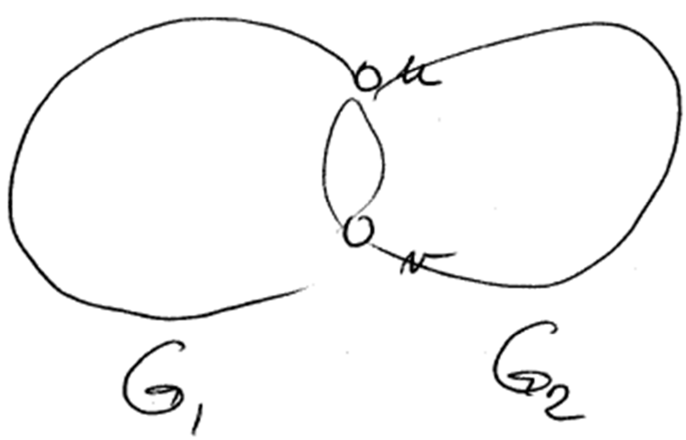
□

Lemma 2.13. Let G have ≥ 4 vertices and no K_5 or $K_{3,3}$ subdivision. Assume that adding an edge joining a pair of non-adjacent vertices creates a K_5 or $K_{3,3}$ subdivision. Then G is 3-connected.

Proof. Induction on $|V(G)|$.

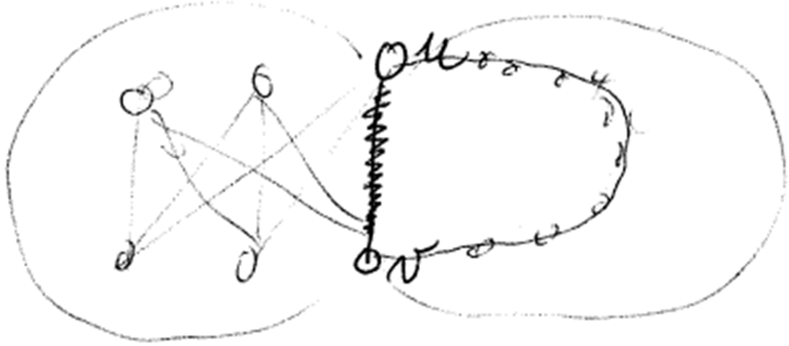
Exercise. Prove G is 2-connected.

To show G is 3-connected suppose that $G\backslash\{u, v\}$ is disconnected. Thus $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{u, v\}$.

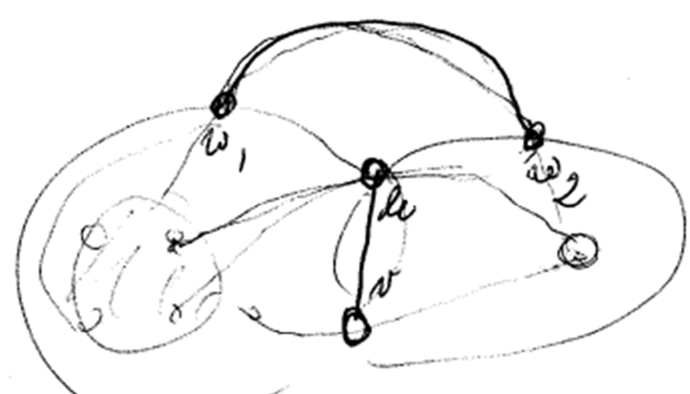


Claim. $u \sim v$.

Pf. If not, then the edge uv can be added without creating K_5 or $K_{3,3}$ subdivision.



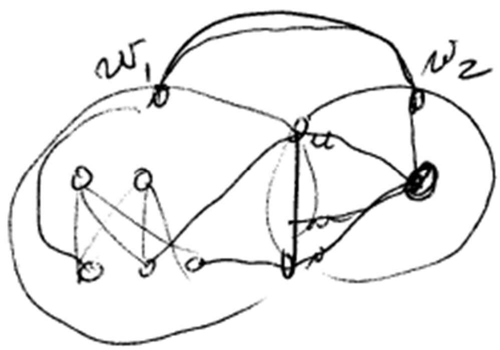
So $u \sim v$. WMA $uv \in E(G_1) \cap E(G_2)$. By induction both G_1, G_2 are either 3-connected or a triangle. By Lemma 2.10 WMA G_i is a plane graph. Pick $w_i \in V(G_i) - \{u, v\}$ such that u, v, w_i belong to the same face. By hypothesis $G + w_1 w_2$ has a K_5 or $K_{3,3}$ subdivision, say K.



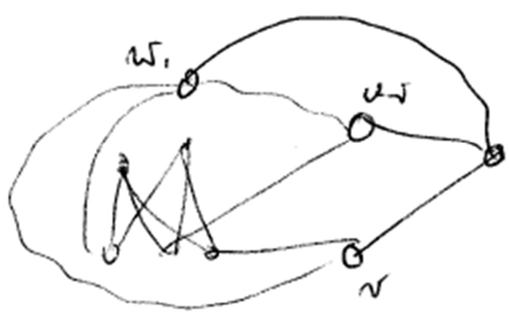
All except possibly one of the branch-vertices of K belong to the same G_i, say G_1. It follows that some branch-vertex of K belongs to G_2, for otherwise there would be a K_5 or $K_{3,3}$ subdivision in G_1. Thus K is a $K_{3,3}$ subdivision.

Let $G_1' := G_1 +$ new vertex adjacent to u, v, w_1

Then G_1' is a plane graph. K can be converted to a $K_{3,3}$ subdivision in G_1', a contradiction.



$G + w_1 w_2$



G_1'

Uniqueness of planar drawings

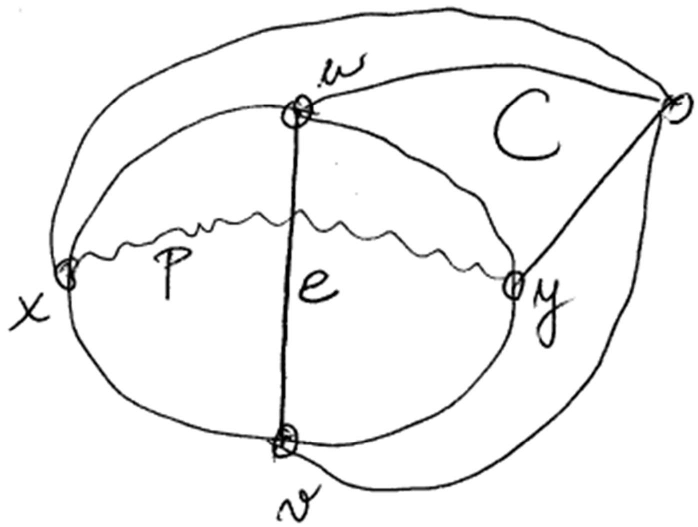
Def. A cycle C in a graph G is **peripheral** if it is induced and $G\backslash V(C)$ has at most one component.

Theorem. Let G be a 3-connected simple plane graph and let C be a subgraph of G. Then C bounds a face of G if and only if C is a peripheral cycle.

Corollary. Every 3-connected simple planar graph G has a unique planar drawing in the sense that every two planar drawings of G have the same facial boundaries.

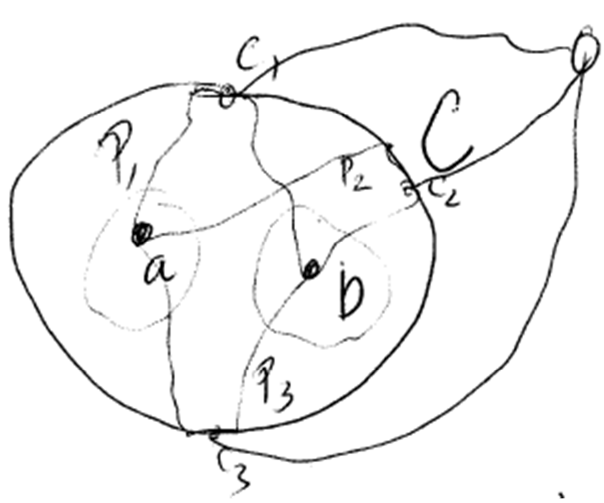
Proof. $\Leftarrow$ Let C be a peripheral cycle. Then by the Jordan curve theorem one of the faces of C, say f, is disjoint from G and its boundary is C. By Lemma 1.10 f is a face of G, and hence C bounds a face of G.

⇒ Let C be the boundary of a face of $f \in F(G)$. Then C is a cycle by Lemma 2.4. Suppose $\exists\ e = uv \in E(G) - E(C)$ with $u, v \in V(C)$.



$\exists$ path from x to y and otherwise disjoint from C, where x, y belong to different components of $C\backslash\{u, v\}$. Add a new vertex in f joined to u, v, x, y. The new graph is planar, but has a K_5 subdivision, a contradiction. So uv does not exist.

Suppose $G\backslash V(C)$ has ≥ 2 components, and let $a, b \in V(G) - V(C)$ belong to different components of $G\backslash V(C)$.



By Menger’s theorem ∃ internally disjoint a- b paths P_1, P_2, P_3. Pick $c_i \in V(P_i) \cap V(C)$ for $i = 1,2,3$. Add a new vertex in f joined to c_1, c_2, c_3. This gives a plane graph with a subdivision of $K_{3,3}$, a contradiction. □