Application to geometry

$cr(G) :=$ minimum number of crossings in a drawing of G in the plane (in which crossings are allowed). More precisely, our "drawings" now allow edges to intersect, but

(1) $|e \cap e'|$ is finite for distinct $e, e' \in E(G)$
(2) each point of $\mathbb{R}^2$ belongs to ≤ 2 edges.

So the number of crossings in a drawing is $\sum_{\{e,e'\}} |e \cap e'|$, and $cr(G)$ is the minimum, over all drawings Γ of G, of the number of crossings in Γ.

Examples. $cr(K_5) = cr(K_{3,3}) = 1$, $cr(K_6) = 3$ (exercise)

Fact. Computing $cr(G)$ is NP-hard.

Conjecture. $cr(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$

Remark. Let $cr'(G)$ denote the minimum, over all drawings Γ of G, of the number of unordered pairs of edges that cross in Γ. It is not known whether $cr(G) = cr'(G)$ for all graphs G.

Lemma. $cr(G) \geq |E(G)| - 3|V(G)|$.

Proof. If not, then remove $cr(G)$ edges, one from each crossing, to get a planar graph on n vertices and $\geq |E(G)| - cr(G) > 3n$ edges, a contradiction.

Crossing Number Lemma. (Ajtai, Chvatal, Newborn, Szemerédi; Leighton) Let G be a simple graph. Then

$$cr(G) \geq \frac{1}{64}\frac{|E(G)|^3}{|V(G)|^2} - |V(G)|$$

Proof. (Székely) Let c be the crossing number of G, let $n = |V(G)|$, $m = |E(G)|$. WMA $m \geq 4n$, for o.w. RHS is negative. Let $p \in (0,1)$, TBD. Choose a random subset $V \subseteq V(G)$ by picking each vertex independently at random with probability p. The expected number of

vertices is pn

edges is p^2m

crossings is p^4c

By the lemma

$$p^4c \geq p^2m - 3pn$$

and so

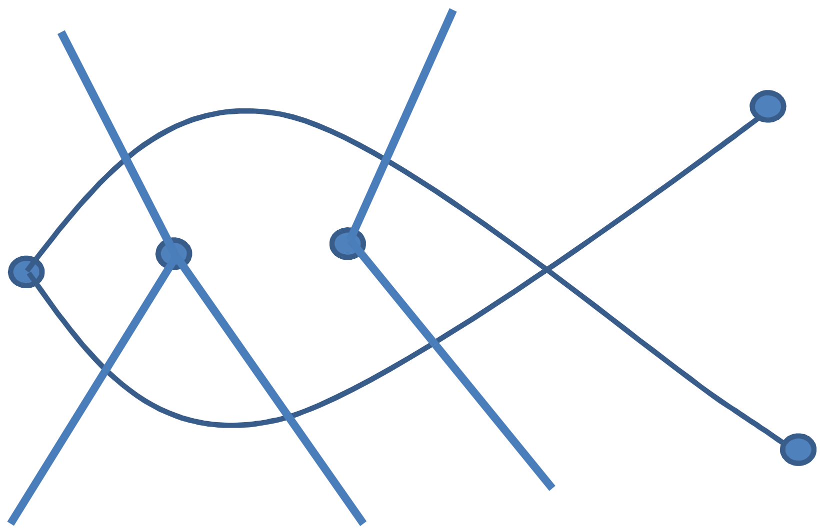
$$c \geq \frac{m}{p^2} - 3\frac{n}{p^3}$$

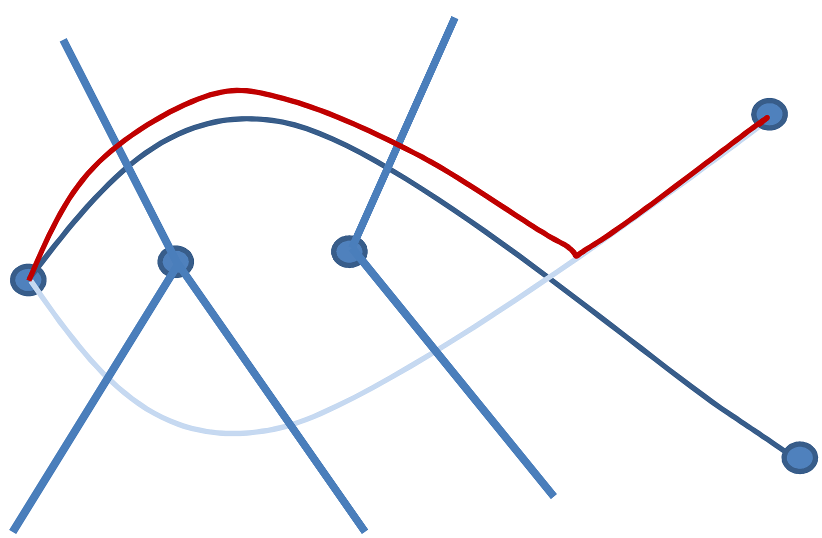
Choose $p = \frac{4n}{m}$ (which is < 1, because $m \geq 4n$) to get

$$c \geq \frac{m^3}{64n^2}$$

$\square$

Is the expected number of crossings really $p^4 c$?



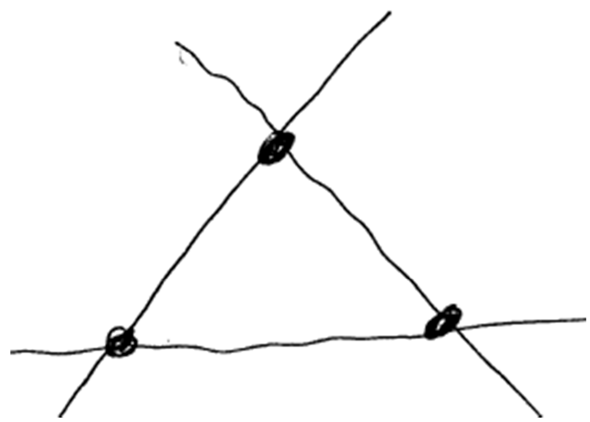


Let $I(n, m)$ be the maximum number of possible incidences between n points and m lines in the plane. That is,

$$I(n, m) = \max|\{(p, L): p \in P, L \in \mathcal{L}, p \in L\}|$$

where the maximum is taken over all sets $P \subseteq \mathbb{R}^2$ and sets of lines $\mathcal{L}$ such that $|P| = n$ and $|\mathcal{L}| = m$.

Example. $I(3,3) \geq 6$

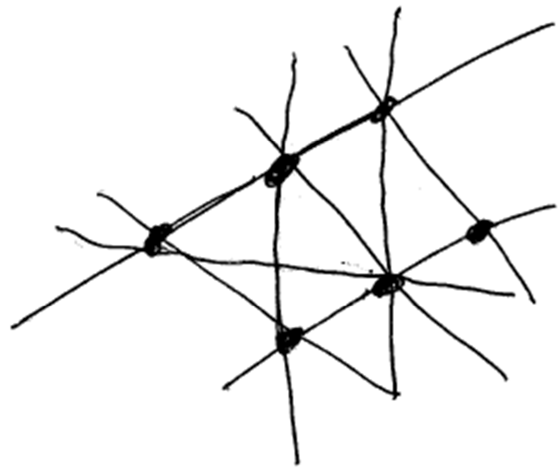


Theorem. (Szemerédi-Trotter) For all $m, n \geq 1$,

$$I(n, m) = O(n^{2/3} m^{2/3} + n + m)$$

and the bound is asymptotically tight.

Proof. (Szekély) Let $P, \mathcal{L}$ be a system of points and lines realizing $I(n, m)$. Define a topological graph (= graph drawn with crossings) G by $V(G) = P$ and $E(G) =$ subsets of lines in $\mathcal{L}$ connecting consecutive points.



A line $L \in \mathcal{L}$ containing k points contributes $k-1$ edges. So

$$I(n,m) = \sum_{L \in \mathcal{L}} \#\text{ of points on } L =$$

$$= \sum_{L \in \mathcal{L}} (1 + \#\text{ edges of } G \text{ contributed by } L) = |E(G)| + m.$$

By the Crossing Number Lemma

$$\frac{1}{2}m^2 \geq \binom{m}{2} \geq cr(G) \geq \frac{1}{64}\frac{|E(G)|^3}{n^2} - n$$

$$|E(G)|^3 \leq 32m^2n^2 + 64n^3$$

$$I(n,m) = |E(G)| + m \leq O(m^{2/3}n^{2/3} + n + m)$$

□