Algebraic graph theory

The **edge space** of a graph G is the vector space $\mathcal{E}(G) \coloneqq GF(2)^{E(G)}$
If $E(G) = \{e_1, e_2, \dots, e_m\}$, then the elements are m-tuples $(x_1, x_2, \dots, x_m)$, where $x_i \in \{0,1\}$.

We can also regard the elements of $\mathcal{E}(G)$ as subsets of $E(G)$ with addition defined by $A + B \coloneqq A \triangle B = (A - B) \cup (B - A)$.

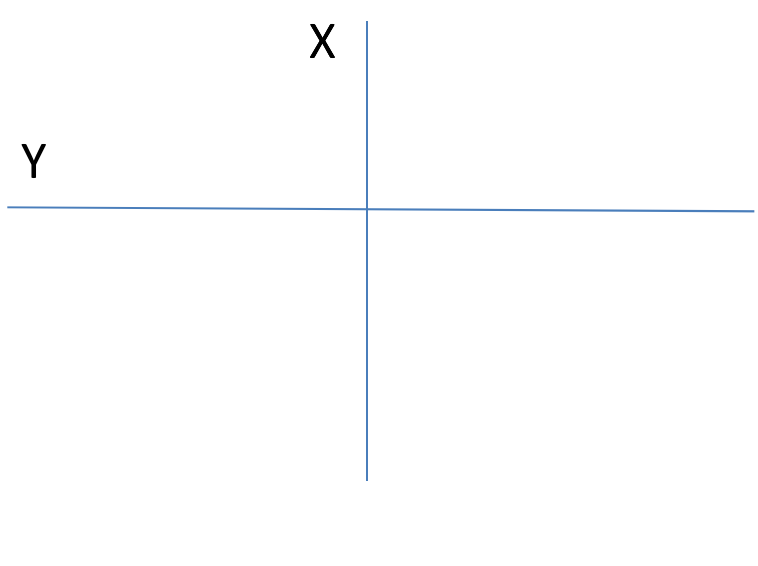
Definition. The **cycle space** of G is the subspace of $\mathcal{E}(G)$ generated by edge-sets of cycles. It is denoted by $\mathcal{C}(G)$.

Proposition. $F \in \mathcal{C}(G)$ if and only if every vertex of G is incident with an even number of edges in F.

Definition. The **cut space** of G is the subspace of $\mathcal{E}(G)$ generated by cuts; that is, sets of edges of the form $\delta(X)$ for some $X \subseteq V(G)$. It is denoted by $\mathcal{C}^*(G)$.

Proposition. $F \in \mathcal{C}^*(G)$ if and only $F = \delta(X)$ for some $X \subseteq V(G)$.

Proof. $\delta(X) \triangle \delta(Y) = \delta(X \triangle Y)$



Definition. For $E, F \in \mathcal{E}(G)$ we define

$$\langle E, F\rangle = |E \cap F| (\text{mod } 2)$$

For $\mathcal{F} \subseteq \mathcal{E}(G)$ let $\mathcal{F}^{\perp} := \{D \in \mathcal{E}(G) : \langle D, F\rangle = 0 \ \forall \ F \in \mathcal{F}\}$

Theorem. $\mathcal{C} = (\mathcal{C}^*)^{\perp}$ and $\mathcal{C}^{\perp} = \mathcal{C}^*$.

Proof. Enough to prove the first equality. $\mathcal{C} \subseteq (\mathcal{C}^*)^{\perp}$, because every cycle intersects every cut even number of times. Conversely, suppose $F \notin \mathcal{C}$. Then $\exists v$ incident with odd number of edges in F. Then $\langle \delta(v), F\rangle = 1$, and so $F \notin (\mathcal{C}^*)^{\perp}$.

Theorem. If G is connected, $n = |V(G)|, m = |E(G)|$, then

$$\dim \mathcal{C}(G) = m - n + 1$$

$$\dim \mathcal{C}^*(G) = n - 1$$

Proof. Pick a spanning tree T. For $e \notin E(T)$ let C_e be the unique cycle in $T + e$. Then $\{C_e\}_{e \notin E(T)}$ are linearly independent. Let $v_0 \in V(G)$. Then

$$\{\delta(v) : v \in V(G) - \{v_0\}\}$$

are linearly independent. Thus the dimensions are at least as large as stated. They sum up to n by the previous theorem. □

Theorem. A graph G is bipartite if and only if it has no odd cycle.

Proof. $\forall$ cycle is even $\Leftrightarrow \mathbb{1} = (1,1, \dots ,1) \in \mathcal{C}(G)^{\perp} = \mathcal{C}^*(G) \Leftrightarrow \exists$ cut that contains all edges $\Leftrightarrow G$ is bipartite. □

Result. Let G be a connected graph with n vertices and m edges. Then G has exactly 2^{m-n+1} even subgraphs (subgraphs with all degrees even).

Proof. The number of such subgraphs is the number of elements in $\mathcal{C}(G)$, which is

$$2^{\dim \mathcal{C}(G)} = 2^{m-n+1}$$

Let A be a symmetric real matrix. The **numerical range** of A is

$$R = \{\langle A\bar{x}, \bar{x}\rangle : \|\bar{x}\| = 1\}$$

Then the largest eigenvalue is $\lambda_{\max} = \max R$ and the smallest eigenvalue is $\lambda_{\min} = \min R$.

There is an orthogonal basis of eigenvectors, and so the 2nd smallest eigenvalue is

$$\min\{\langle A\bar{x}, \bar{x}\rangle : \|\bar{x}\| = 1, \langle \bar{x}, \bar{x}_1\rangle = 0\}$$

where $\bar{x}_1$ is an eigenvector corresponding to $\lambda_{\min}$.

Theorem. Let G be a connected graph with adjacency matrix A. Then

(i) $|\lambda| \leq \Delta(G)$ for every eigenvalue λ of G

(ii) $\Delta(G)$ is an eigenvalue of A if and only if G is regular, and if it is, then $\Delta(G)$ has multiplicity 1

(iii) if $-\Delta(G)$ is an eigenvalue, then G is regular and bipartite

(iv) if G is bipartite and λ is an eigenvalue of A, then so is $-\lambda$ and they have the same multiplicity

(v) the largest eigenvalue λ satisfies $\delta(G) \leq \lambda \leq \Delta(G)$

(vi) if H is an induced subgraph of G, then

$$\lambda_{\min}(G) \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq \lambda_{\max}(G)$$

Proof. (i) Let $\bar{x}$ be an eigenvector corresponding to λ. By reordering $V(G)$ WMA $1 = |x_1| \geq |x_2|, |x_3|, \dots, |x_n|$. Then

$$|\lambda| = |\lambda x_1| = |(A\bar{x})_1| = \left|\sum_{j=1}^{n} A_{1j}x_j\right| \leq$$

$$\leq \sum_{j=1}^{n} A_{1j}|x_j| \leq \sum_{j=1}^{n} A_{1j} = \deg(v_1) \leq \Delta(G)$$

(ii) If G is regular, then $\mathbb{1} = (1,1,\dots,1)$ is an eigenvector corresponding to $\Delta(G)$. Conversely, if $\Delta(G)$ is an eigenvalue, then the above calculation (without absolute values) shows that G is regular and $\mathbb{1}$ is the only eigenvector with $x_1 = 1$.

(iii) If $-\Delta(G)$ is an eigenvalue, then the same argument shows G is regular. Suppose $x_1 = -1$. Then

$$\Delta(G) = -\Delta(G)x_1 = (A\bar{x})_1 = \sum_{j=1}^{n} A_{1j}x_j \leq \Delta(G)$$

$\Rightarrow x_j = 1$ for every neighbor v_j of v_1.

The same argument shows that $x_k = -1$ for every neighbor x_k of a neighbor of v_1, and so on. Thus G is bipartite.

(iv) $A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$. Let $\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$ be an eigenvector corresponding to λ. Then $B\bar{x}_2 = \lambda\bar{x}_1$, $B^T\bar{x}_1 = \lambda\bar{x}_2$. Let $\bar{y} \coloneqq \begin{pmatrix} \bar{x}_1 \\ -\bar{x}_2 \end{pmatrix}$. Then $A\bar{y} = \begin{pmatrix} -B\bar{x}_2 \\ B^T\bar{x}_1 \end{pmatrix} = \begin{pmatrix} -\lambda\bar{x}_1 \\ \lambda\bar{x}_2 \end{pmatrix} = -\lambda\bar{y}$.

(v) $\lambda_{\max} = \max\{\langle A\bar{x}, \bar{x}\rangle : \|\bar{x}\| = 1\}$. We know $\lambda_{\max} \leq \Delta(G)$. Now take $\bar{x} = \frac{1}{\sqrt{n}}(1,1, \dots ,1)$. Then

$$\lambda_{\max} \geq \langle A\bar{x}, \bar{x}\rangle = \frac{1}{n}\sum_{(i,j)} A_{ij} \geq \frac{1}{n}\sum_{v\in V(G)} \deg(v) \geq \delta(G)$$

(vi) Enough to show for $H \coloneqq G\backslash v_n$. Pick $\bar{y}$ with $\|\bar{y}\| = 1$ such that $\langle A'\bar{y}, \bar{y}\rangle = \lambda_{\max}(H)$, where $A' =$ adj matrix of H. Let $\bar{x} = (y_1, y_2, \dots , y_{n-1}, 0)$. Then $\|\bar{x}\| = 1$ and

$$\lambda_{\max}(G) \geq \langle A\bar{x}, \bar{x}\rangle = \langle A'\bar{y}, \bar{y}\rangle = \lambda_{\max}(H)$$

The other inequality is analogous. $\square$

Reminder: The Laplace matrix L of a graph

$$L_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -A_{ij} & \text{if } i \neq j \end{cases}$$

Exercise. $\langle L\bar{x}, \bar{x}\rangle = \sum_{\substack{\{i,j\} \\ i\sim j}}(x_i - x_j)^2$ for any vector $\bar{x}$, where L is the Laplace matrix of G.

Theorem. Let G be a graph on n vertices, and let λ_2 be the second smallest eigenvalue of the Laplace matrix of G. Then for every $S \subseteq V(G)$ we have

$$|\partial S| \geq \lambda_2 \frac{|S| \cdot |V(G) - S|}{n}$$

Proof. $\lambda_2 = \min\{\langle L\bar{x}, \bar{x}\rangle : \|\bar{x}\| = 1, \bar{x} \cdot \mathbb{1} = 0\}$. Let $|S| = k$. Define $\bar{x} = (x_1, \dots x_n)$ by

$$x_i = \begin{cases} n-k & \text{if } v_i \in S \\ -k & \text{if } v_i \notin S \end{cases}$$

$\bar{x} \cdot \mathbb{1} = k(n-k) - k(n-k) = 0$

$\|\bar{x}\|^2 = k(n-k)^2 + (n-k)k^2 = nk(n-k)$

$$\lambda_2 \leq \frac{\langle L\bar{x}, \bar{x} \rangle}{\|\bar{x}\|^2} = \frac{\sum_{\{i,j\}, i\sim j} (x_i - x_j)^2}{nk(n-k)} = \frac{|\partial S| n^2}{nk(n-k)}$$

$\square$

Definition. The conductance or **isoperimetric constant** of G is

$$\Phi(G) := \min \left\{ \frac{|\partial S|}{|S|} : \emptyset \neq S \subseteq V(G), |S| \leq \frac{|V(G)|}{2} \right\}$$

Corollary. $\Phi(G) \geq \lambda_2(G)/2$.

Definition. A graph G is an $(\boldsymbol{n}, \boldsymbol{d}, \boldsymbol{c})$**-expander** if $|V(G) = n$, $\Delta(G) \leq d$ and for every $X \subseteq V(G)$ with $|X| \leq n/2$ we have $|N(X)| \geq c|X|$.

$N(X)$ means neighbors outside of X

Corollary. Every graph G on n vertices and $\Delta(G) \leq d$ is an (n, d, c)-expander, where $c = \frac{\lambda_2}{2d}$ and λ_2 is the second smallest eigenvalue of the Laplace matrix of G.

Proof. $|N(X)| \geq \frac{1}{d} |\partial X| \geq \frac{\lambda_2}{2d} |X|$ by the previous corollary.

Theorem (Alon 1986) If a graph G is an (n, d, c)-expander, then

$$\lambda_2 \geq \frac{c^2}{4+2c^2}$$

I will not prove this, but I have notes on it.

Szemerédi's regularity lemma.

Let $\varepsilon > 0$. Let $A, B \subseteq V(G)$ be disjoint. We say that (A, B) is ε-**regular** in G if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have

$$|d(A,B) - d(X,Y)| \leq \varepsilon$$

where

$$d(C,D) = \frac{|\langle C,D\rangle|}{|C| \cdot |D|}$$

and $\langle C,D\rangle = \{e : e \text{ has one end in } C, \text{the other in } D\}$.

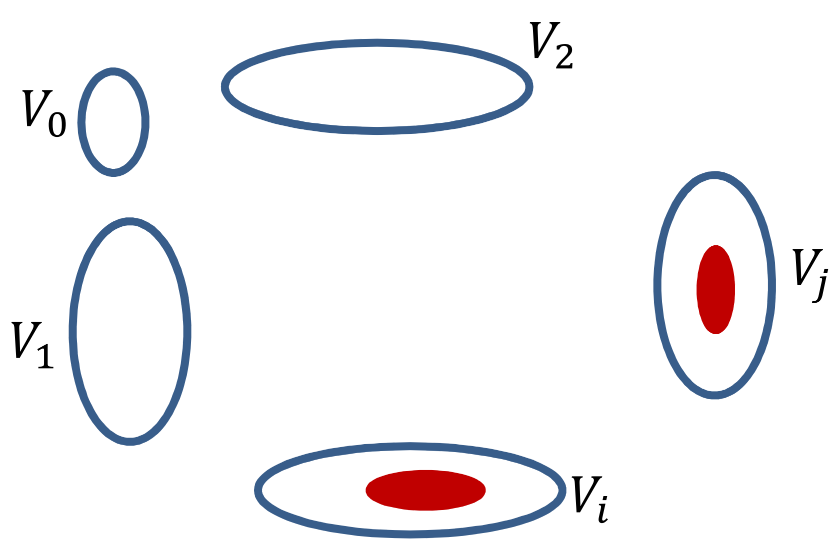
A partition $(V_0, V_1, \dots, V_k)$ of $V(G)$ is $\boldsymbol{\varepsilon}$**-regular** if

(i) $|V_0| \leq \varepsilon|V(G)|$

(ii) $|V_1| = |V_2| = \cdots = |V_k|$

(iii) all but at most εk^2 pairs (V_i, V_j) are ε-regular $(1 \leq i, j \leq k)$.

Theorem (Szemerédi's regularity lemma). $\forall\, \varepsilon > 0\ \forall$ integer $m\ \exists$ integer $M\ \forall$ graph G on $n \geq m$ vertices $\exists\, k$ with $m \leq k \leq M$ and an ε-regular partition $(V_0, V_1, V_2, \dots, V_k)$ of $V(G)$.



The Erdős-Stone theorem

Turán's theorem. If G has no K_r subgraph, then

$$|E(G)| \leq \left|E\left(T_{r-1}(n)\right)\right|,$$

with equality if and only if $G \cong T_{r-1}(n)$.

Recall that $T_{r-1}(n) =$ complete $(r-1)$-partite graph on n vertices with color classes as close to each other in size as possible. Let

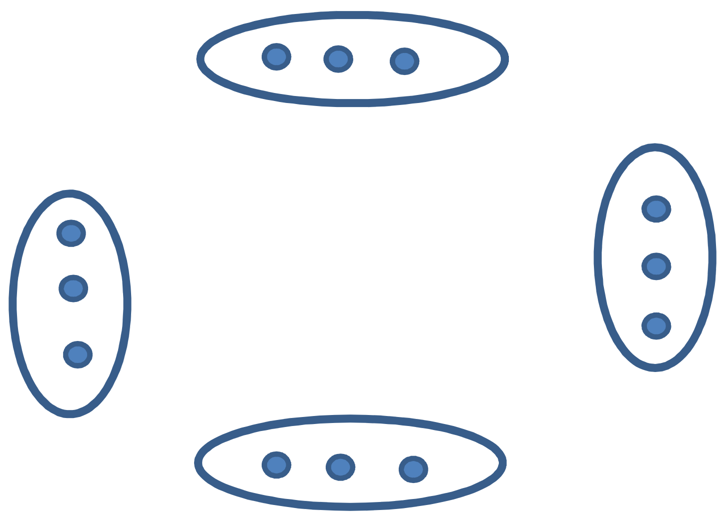
$$t_{r-1}(n) \coloneqq \left|E\left(T_{r-1}(n)\right)\right| \approx \frac{r-2}{r-1} \cdot \frac{n^2}{2}$$

Theorem. (Erdős-Stone) $\forall\, r, s\ \forall\, \varepsilon > 0\ \exists n_0 \forall$ graph G on $\geq n_0$ vertices if

$$|E(G)| \geq t_{r-1}(n) + \varepsilon n^2,$$

then G has a $K_s^r \coloneqq K_{s,s,\ldots,s}$-subgraph.

$K_3^4 =$



Equivalently, the hypothesis can be stated as

$$|E(G)| \geq \left(\frac{r-2}{r-1} + \varepsilon\right)\frac{n^2}{2}$$

Definition.

$$\operatorname{ex}(n, H) \coloneqq \max\{|E(G)| : |V(G)| = n,\ G \text{ has no } H \text{ subgraph}\}$$

Example. $\operatorname{ex}(n, K_r) = t_{r-1}(n)$.

Corollary. $\lim_{n\to\infty} \frac{\operatorname{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$ for every H with ≥ 1 edge

Proof. Weekly exercise.

Definition. The upper density of an (infinite) graph G is

$$\limsup \left\{ \frac{|E(H)|}{\binom{|V(H)|}{2}} : H \subseteq G, H \text{ finite} \right\}$$

Corollary. The upper density of an infinite graph is

$$0, {}^{1}\!/_{2}, {}^{2}\!/_{3}, {}^{3}\!/_{4}, \dots, \text{or } 1$$

Proof. Weekly exercise.