Szemerédi's regularity lemma.

Let $\varepsilon > 0$. Let $A, B \subseteq V(G)$ be disjoint. We say that (A, B) is ε-**regular** in G if for all $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have

$$|d(A,B) - d(X,Y)| \leq \varepsilon$$

where

$$d(C,D) = \frac{|\langle C,D\rangle|}{|C| \cdot |D|}$$

and $\langle C,D\rangle = \{e : e \text{ has one end in } C, \text{the other in } D\}$.

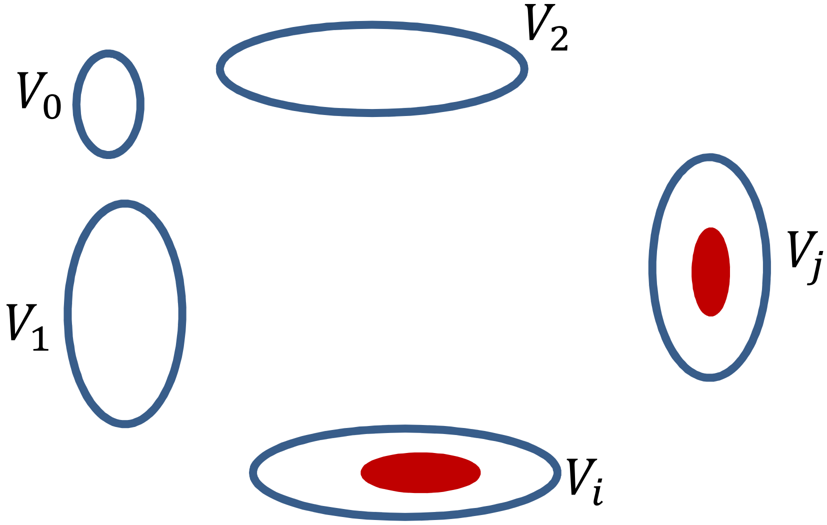
A partition $(V_0, V_1, \dots, V_k)$ of $V(G)$ is $\boldsymbol{\varepsilon}$**-regular** if

(i) $|V_0| \leq \varepsilon|V(G)|$

(ii) $|V_1| = |V_2| = \cdots = |V_k|$

(iii) all but at most εk^2 pairs (V_i, V_j) are ε-regular $(1 \leq i, j \leq k)$.

Theorem (Szemerédi's regularity lemma). $\forall\, \varepsilon > 0\ \forall$ integer $m\ \exists$ integer $M\ \forall$ graph G on $n \geq m$ vertices $\exists\, k$ with $m \leq k \leq M$ and an ε-regular partition $(V_0, V_1, V_2, \dots, V_k)$ of $V(G)$.



The Erdős-Stone theorem

Turán's theorem. If G has no K_r subgraph, then

$$|E(G)| \leq \left|E\left(T_{r-1}(n)\right)\right|,$$

with equality if and only if $G \cong T_{r-1}(n)$.

Recall that $T_{r-1}(n)$ = complete $(r-1)$-partite graph on n vertices with color classes as close to each other in size as possible. Let

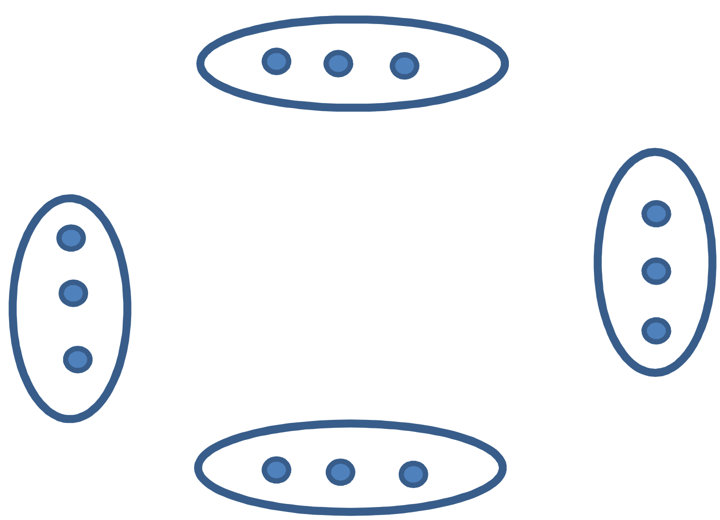
$$t_{r-1}(n) \coloneqq \left|E\left(T_{r-1}(n)\right)\right| \approx \frac{r-2}{r-1} \cdot \frac{n^2}{2}$$

Theorem. (Erdős-Stone) $\forall\, r, s\ \forall\, \varepsilon > 0\ \exists n_0 \forall$ graph G on $\geq n_0$ vertices if

$$|E(G)| \geq t_{r-1}(n) + \varepsilon n^2,$$

then G has a $K_s^r \coloneqq K_{s,s,\dots,s}$-subgraph.

$K_3^4 =$



Equivalently, the hypothesis can be stated as

$$|E(G)| \geq \left(\frac{r-2}{r-1} + \varepsilon\right)\frac{n^2}{2}$$

Definition.

$$\text{ex}(n,H) := \max\{|E(G)| : |V(G)| = n,\ G \text{ has no } H \text{ subgraph}\}$$

Example. $\text{ex}(n, K_r) = t_{r-1}(n)$.

Corollary. $\lim_{n\to\infty} \frac{\text{ex}(n,H)}{\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}$ for every H with ≥ 1 edge

Proof. Weekly exercise.

Definition. The upper density of an (infinite) graph G is

$$\limsup \left\{ \frac{|E(H)|}{\binom{|V(H)|}{2}} : H \subseteq G, H \text{ finite} \right\}$$

Corollary. The upper density of an infinite graph is

$$0, {}^{1}/_{2}, {}^{2}/_{3}, {}^{3}/_{4}, \ldots, \text{or } 1$$

Proof. Weekly exercise.

Proof of Erdős-Stone. Let r, s and $\gamma > 0$ be given. Want to show $\exists n_0$ such that if $|V(G)| \geq n_0$, then

$$|E(G)| \geq \left(\frac{1}{2}\frac{r-1}{r-2} + \gamma\right) n^2 \Rightarrow K_s^r \subseteq G$$

Let $\varepsilon = \varepsilon(r, s, \gamma)$ and $m = m(\gamma)$, TBD.

Let M be as in Szemerédi's regularity lemma. Let G have at least m vertices. By Szemerédi's regularity lemma G has an ε-regular partition $(V_0, V_1, \dots, V_k)$ with $m \leq k \leq M$.

Note $(1-\varepsilon)\frac{n}{k} \leq |V_i| \leq \frac{n}{k}$ for $i = 1,2, \dots, k$.

Let R be the "regularity graph" defined by $V(R) \coloneqq \{1,2, \dots, k\}$ and $i \sim j$ in R if (V_i, V_j) is ε-regular and the density $d(V_i, V_j)$ is $\geq \gamma$.

Claim 1. ε, m can be chosen so that R has a K_r subgraph.

Pf. If not, then by Turán's theorem $|E(R)| \leq t_{r-1}(k) \leq \frac{1}{2}\frac{r-2}{r-1}k^2$.

We have that $|E(G)|$ is at most the sum of:

- # of edges incident with V_0, which is $\leq \varepsilon n^2$
- # edges in some V_i , which is $\leq k\frac{1}{2}\left(\frac{n}{k}\right)^2 \leq \frac{1}{2}\frac{n^2}{m}$
- # of V_i-V_j edges for $i, j \in E(R)$, which is $\leq |E(R)|\left(\frac{n}{k}\right)^2 \leq \frac{1}{2}\frac{r-2}{r-1}n^2$
- # of V_i-V_j edges for (V_i, V_j) not ε-regular, which is $\leq \varepsilon k^2\left(\frac{n}{k}\right)^2$
- # of V_i-V_j edges when $d(V_i, V_j) < \gamma$, which is $\leq \binom{k}{2}\gamma\left(\frac{n}{k}\right)^2$

$$|E(G)| \le \varepsilon n^2 + k\frac{1}{2}\left(\frac{n}{k}\right)^2 + |E(R)|\left(\frac{n}{k}\right)^2 + \varepsilon k^2\left(\frac{n}{k}\right)^2 + \binom{k}{2}\gamma\left(\frac{n}{k}\right)^2$$

$$\le \frac{1}{2}\frac{r-2}{r-1}n^2 + \varepsilon n^2 + \frac{1}{2}\frac{n^2}{k} + \varepsilon n^2 + \frac{\gamma}{2}n^2 \le$$

$$\le \frac{1}{2}\frac{r-2}{r-1}n^2 + \left(2\varepsilon + \frac{1}{2m} + \frac{\gamma}{2}\right)n^2$$

So on choosing ε, m such that $4\varepsilon + \frac{1}{m} < \gamma$ we get a contradiction. This proves Claim 1.

Next we show that if ε is sufficiently small, then $K_s^r \subseteq G$. We may assume that $\{1,2,\dots,r\}$ is a clique in R.

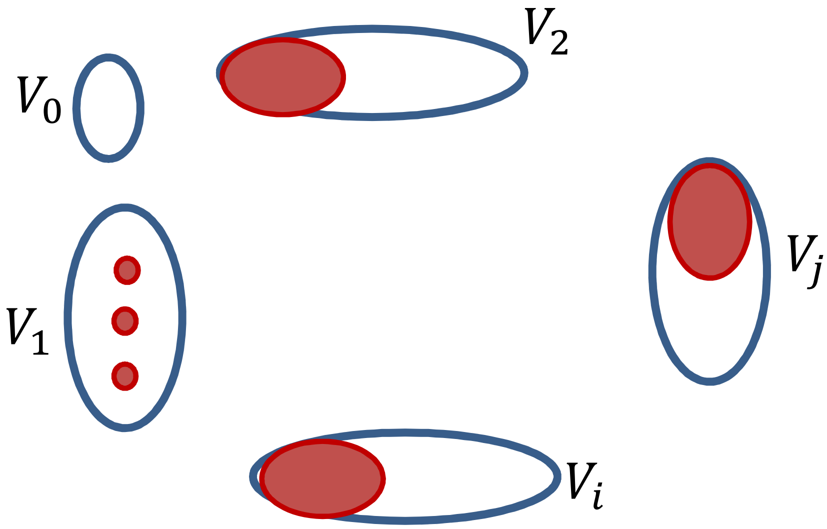
For $1 \le i < j \le r$ we have $d(V_i, V_j) \ge \gamma$ and so $d(X,Y) \ge \gamma - \varepsilon$ for every $X \subseteq V_i$ and $Y \subseteq V_j$ with $|X| \ge \varepsilon |V_i|$ and $|Y| \ge \varepsilon |V_j|$.

We will show:

(*) Let $V_i' \subseteq V_i$ satisfy $|V_i'| \ge \varepsilon |V_i|$ and $|V_1'| \ge (r-1)\varepsilon |V_1| + s$. Then there exist distinct vertices $v_1, v_2, \dots, v_s \in V_1'$ and sets $V_i'' \subseteq V_i'$ such that

- $\{v_1, \dots, v_s\}$ is complete to V_i'' for all $i = 2, \dots, r$
- $|V_i''| \ge c|V_i'|$, where $c = c(\gamma, r)$ is independent of ε

To prove the theorem assuming (*), first apply (*) to $V_i' = V_i$ to obtain $v_1, v_2, \dots, v_s \in V_1'$ and sets $V_i' \subseteq V_i$. Then apply (*) to the sets $V_2', V_3', \dots, V_r'$ to obtain $u_1, u_2, \dots, u_s \in V_2'$ and sets $V_i'' \subseteq V_i'$. Then apply (*) to the sets $V_3'', V_4'', \dots, V_r''$ and so on.



Lemma. Let $Y \subseteq V_2$ with $|Y| \geq \varepsilon|V_2|$. Then all but $\varepsilon|V_1|$ vertices of V_1 have $\geq (\gamma - \varepsilon)|Y|$ neighbors in Y.

Proof. Let

$$X = \{v \in V_1 : v \text{ is adjacent to } < (\gamma - \varepsilon)|Y| \text{ vertices in } Y\}.$$

Then $|\langle X, Y \rangle| < |X|(\gamma - \varepsilon)|Y|$. Thus $d(X, Y) < \gamma - \varepsilon$, and so $|X| < \varepsilon|V_1|$, as desired. □

By the lemma, all but $(r-1)\varepsilon|V_1|$ vertices of V_1 have $\geq (\gamma - \varepsilon)|Y|$ neighbors in Y for $Y = V_2', V_3', \dots, V_r'$. Pick $v_1 \in V_1'$ with this property and let $W_i \subseteq V_i'$ be such that $|W_i| \geq (\gamma - \varepsilon)|V_i'|$ and v_1 is complete to W_i for all $i = 2,3, \dots, r$. Repeat the same argument, but with $V_2', V_3', \dots, V_r'$ replaced by $W_2, \dots, W_r$. We find $v_2 \in V_1' - \{v_1\}$ and $Z_i \subseteq W_i$ such that $|Z_i| \geq (\gamma - \varepsilon)|W_i|$ and v_2 is complete to Z_i for all $i = 2,3, \dots, r$. After s iterations we will end up with the vertices $v_1, v_2, \dots, v_s$ and sets $V_2'', V_3'', \dots, V_r''$. □

Theorem (Triangle removal lemma) $\forall \varepsilon > 0 \; \exists \, \delta > 0 \; \exists \, n_0$ such that if G is a graph on $\geq n_0$ vertices with at most δn^3 triangles, then $\exists \, F \subseteq E(G)$ such that $|F| \leq \varepsilon n^2$ and $G \backslash F$ is triangle-free.

"If a graph has $o(n^3)$ triangles, then all triangles can be destroyed by removing $o(n^2)$ edges"

Application to arithmetic progressions

Lemma. $\forall \, \varepsilon > 0 \; \exists \, n_0 \; \forall$ graph G on $n \geq n_0$ vertices such that every edge is in exactly one triangle has $\leq \varepsilon n^2$ edges.

Proof. By the Triangle removal lemma $\exists \, \delta > 0 \; \exists \, n_0$ such that every graph on $\geq n_0$ vertices with $\leq \delta n^3$ triangles can be made Δ-free by deleting εn^2 edges. Our graph has $|E(G)| \leq n^2$ triangles, which is $\leq \delta n^3$ for big enough n. So if n is a big enough, then $\exists \, F \subseteq E(G)$ such that $|F| \leq \varepsilon n^2$ and $G \backslash F$ is triangle free. Let F' consist of the edges of the unique triangle containing e, for all $e \in F$. Thus $|F'| \leq 3\varepsilon n^2$ and $E(G) \subseteq F'$, as required. □

Definition. A **corner** is a triple of the form

$\{(x, y), (x, y + d), (x + d, y)\}$ for some d, possibly negative.

Corollary. (Ajtai & Szemerédi 1974). Let $A \subseteq [\{1,2,\ldots,N\}]^2$. If A contains no corner, then $|A| = o(N^2)$.

Proof. Consider the $N \times N$ grid.

X = horizontal lines $y = i$ for $i = 1,2,\ldots,N$

Y = vertical lines $x = i$ for $i = 1,2,\ldots,N$

Z = slope -1 lines $y = -x + i$ for $i = 1,2,\ldots,2N-1$

Then $|X| = |Y| = N$ and $|Z| = 2N - 1$

Define a tripartite graph G on $X \cup Y \cup Z$ by saying that the two lines are adjacent if their intersection belongs to A. Then $|V(G)| = 4N - 1$ and $|E(G)| = 3|A|$. Since A is corner-free, each edge belongs to a unique triangle. By the previous lemma, $3|A| = o(N^2)$, as desired.

□

Corollary. (Roth's theorem) If $S \subseteq \{1,2,\ldots,N\}$ contains no 3-term arithmetic progression, then $|S| = o(N)$.

Proof. Let $A = \{(x,y): x,y \in \{1,2,\ldots,2N\}, y - x \in S\}$. If A has a corner, say $(x,y), (x,y+d), (x+d,y) \in A$, then

$y - x \in S$, $y + d - x \in S$, $y - x - d \in S$, and so S contains a 3-term arithmetic progression. We may therefore assume that A has no corner. For every $\{s,s'\} \subseteq S$ we have $(s, s+s') \in A$, and hence $|A| \geq \binom{|S|}{2}$. By the previous corollary $\binom{|S|}{2} \leq |A| = o(N^2)$, and so $|S| = o(N)$.

Application to property testing

Definition. A graph G on n vertices is **ε-far from triangle-free** if for every set $F \subseteq E(G)$ of size at most εn^2 the graph $G \backslash F$ has a triangle.

Remark. We cannot hope to test whether a graph is triangle-free in constant time, but how about distinguishing triangle-free graphs from those that are ε-far from triangle-free?

Theorem. For every $\varepsilon > 0$ there exists a randomized algorithm which in constant time accepts every triangle free graph and rejects every graph which is ε-far from triangle-free with probability at least $2/3$.

Proof. Let δ be as in the Triangle removal lemma. Thus if G is ε-far from triangle-free, then it has more than δn^3 triangles. Pick δ^{-1} triples of vertices uniformly independently at random. If none of those triples form a triangle, then accept the graph; otherwise reject. If the graph is a triangle-free, then it will be accepted. If it is ε-far from triangle-free, then the probability of being accepted is at most

$$\left(1 - \frac{\delta n^3}{\binom{n}{3}}\right)^{\delta^{-1}} \leq \frac{1}{3}$$

if δ is sufficiently small.