Nowhere-zero flows

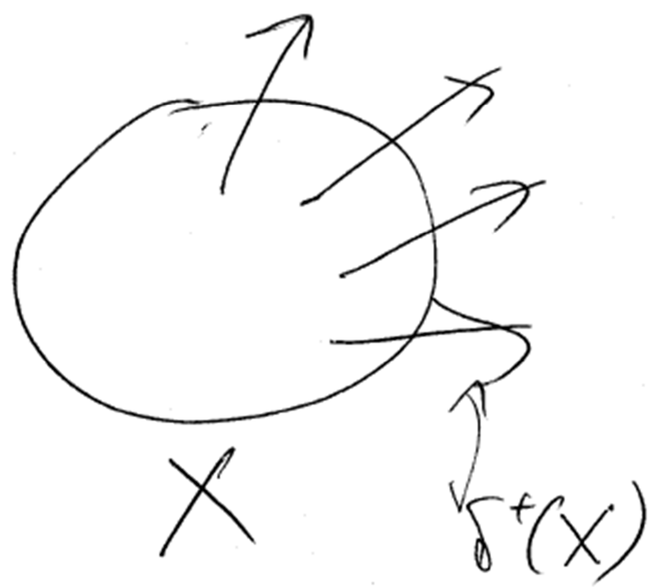
Let D be a digraph, Γ Abelian group. A Γ-circulation in D is a mapping $f: E(D) \to \Gamma$ such that

$$f^+(v) = f^-(v),$$

where $f^+(v) = \sum_{e \in \delta^+} f(e)$, $f^-(v) = \sum_{e \in \delta^-} f(e)$ and

$$\delta^+(X) = \{e \in E(D)\text{: tail in } X\text{, head in } V(D) - X\}$$

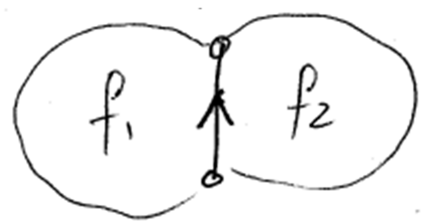
$$\delta^-(X) = \{e \in E(D)\text{: tail in } X\text{, head in } V(D) - X\}$$



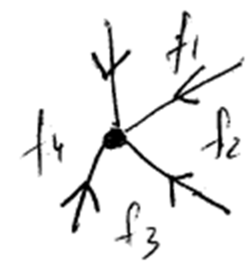
A **nowhere-zero** Γ-flow is a Γ-circulation such that $f(e) \neq 0$ for every $e \in E(D)$. A **nowhere-zero** k-flow is a Z-circulation f such that $0 < |f(e)| < k$ for every edge e. Compare to nowhere-zero (NZ) Z_k-flow. These are properties of the underlying undirected graph of D.

Thm. Let G be a plane graph. Then G has a NZ k-flow if and only if G is face k-colorable.

Pf. $\Leftarrow$ Color the faces using $1, \ldots, k$. Let D be an orientation of G. Define $\phi(e) = c(f_1) - c(f_2)$.



Then $\phi(e) \neq 0 \; \forall \; e \in E(D)$.



$\Rightarrow$ Any integer-valued circulation in a plane graph is an integer linear combination of "facial circulations"

Now let ϕ be a NZ k-flow. Then there exists a function $\beta\colon F(G) \to \mathbb{Z}$ such that

$$\phi(e) = \beta(f_1) - \beta(f_2),$$

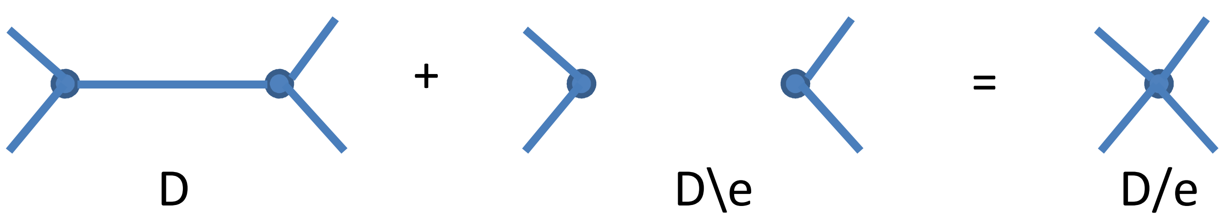
where f_1 is the face to the left of e and f_2 is the face to the right. Define $\alpha(f)$ to be the residue class of $\beta(f)$ (mod k). Then α is a k-coloring of the faces.

Thm. Let D be a digraph. There is a polynomial P such that for every Abelian group Γ the number of NZ Γ-flows in D is $P(|\Gamma|)$.

Proof. If D has no non-loop edge, then $P(x) = (x-1)^{|E(D)|}$.

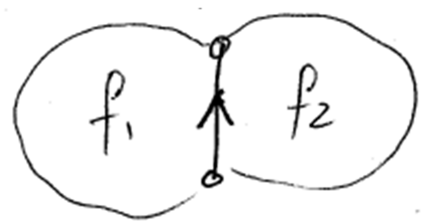
Otherwise pick a non-loop edge e, and let $\phi(D)$ be the # of NZ Γ-flows in D. Then

$$\phi(D) = \phi(D/e) - \phi(D\backslash e)$$

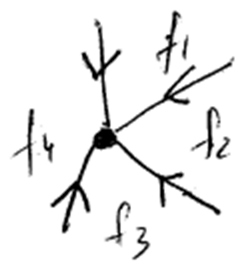


Theorem follows by induction. $\square$

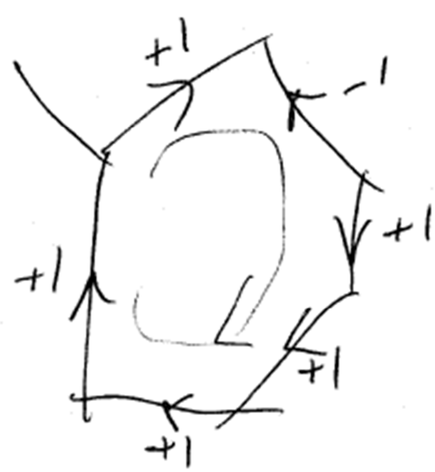
Pf. $\Leftarrow$ Color the faces using $1, \ldots, k$. Let D be an orientation of G. Define $\phi(e) = c(f_1) - c(f_2)$.



Then $\phi(e) \neq 0 \; \forall \; e \in E(D)$.



$\Rightarrow$ Any integer-valued circulation in a plane graph is an integer linear combination of "facial circulations"



Now let ϕ be a NZ k-flow. Then there exists a function $\beta: F(G) \to \mathbb{Z}$ such that

$$\phi(e) = \beta(f_1) - \beta(f_2),$$

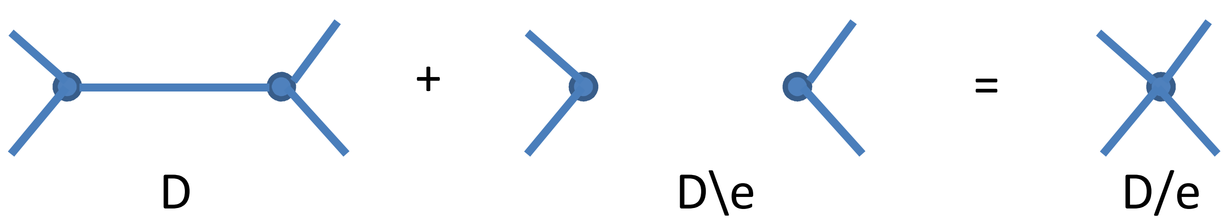
where f_1 is the face to the left of e and f_2 is the face to the right. Define $\alpha(f)$ to be the residue class of $\beta(f)$ (mod k). Then α is a k-coloring of the faces.

Thm. Let D be a digraph. There is a polynomial P such that for every Abelian group Γ the number of NZ Γ-flows in D is $P(|\Gamma|)$.

Proof. If D has no non-loop edge, then $P(x) = (x-1)^{|E(D)|}$.

Otherwise pick a non-loop edge e, and let $\phi(D)$ be the # of NZ Γ-flows in D. Then

$$\phi(D) = \phi(D/e) - \phi(D\backslash e)$$



Theorem follows by induction. □

Thm. A graph has $\mathbb{Z}_k$-flow if and only if it has a k-flow.

Pf. $\Leftarrow$ easy.

$\Rightarrow$ Let $f: E(D) \to \mathbb{Z}$ be such that

(1) $0 < |f(e)| < k \;\forall\; e \in E(D)$

(2) $f^+(v) \equiv f^-(v) (\text{mod } k)$

Let $D(f) \coloneqq \sum_{v \in V(D)} |f^+(v) - f^-(v)|$, and choose f satisfying (1) and (2) with $D(f)$ minimum. WMA $f(e) > 0 \;\forall\; e \in E(D)$. Let

$$A = \{v \in V(D) \colon f^+(v) > f^-(v)\}$$

and

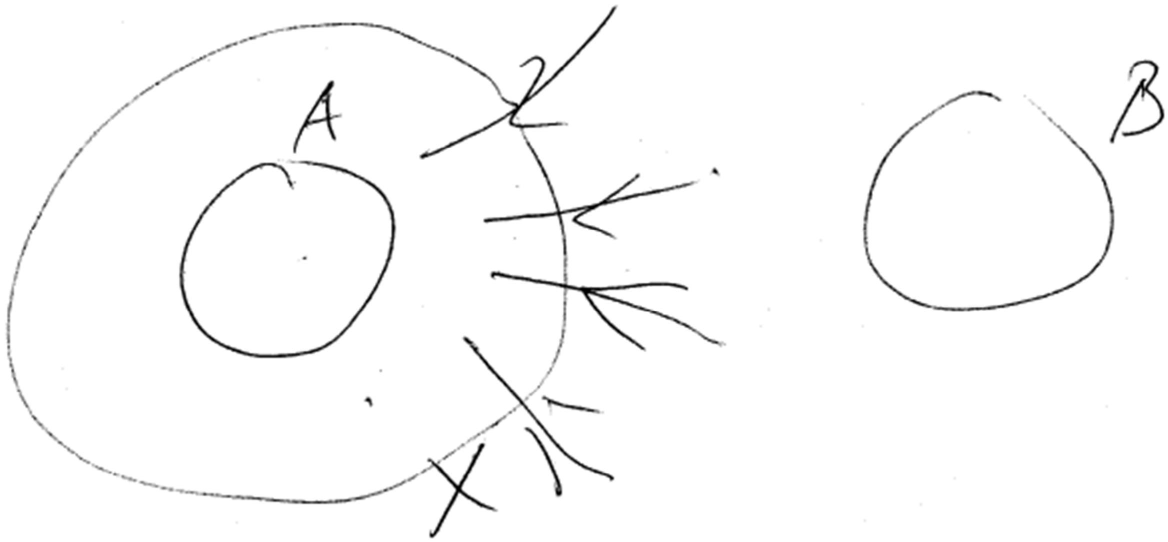
$$B = \{v \in V(D) \colon f^+(v) < f^-(v)\}$$

Claim. $\nexists$ directed $A \to B$ path.

Pf. O.w. decrease the flow by k along such path. If $A = B = \emptyset$, then done, so WMA one is empty, and hence both are, because

$$\sum_{v \in V(D)} f^+(v) = \sum_{e \in E(D)} f(e) = \sum_{v \in V(D)} f^-(v)$$

By the claim $\exists\; X$ with $A \subseteq X$, $B \cap X = \emptyset$ and $\delta^+(X) = \emptyset$.



$$\sum_{v \in X} f^+(v) > \sum_{v \in X} f^-(v)$$

$$-\sum_{e \in \langle X \rangle} f(e) + \sum_{v \in X} f^+(v) > -\sum_{e \in \langle X \rangle} f(e) + \sum_{v \in X} f^-(v)$$

$0 = f^+(X) > f^-(X)$, a contradiction. □

Thm. A graph has $\mathbb{Z}_k$-flow if and only if it has a k-flow.

Pf. $\Leftarrow$ easy.

$\Rightarrow$ Let $f: E(D) \to \mathbb{Z}$ be such that

(1) $0 < |f(e)| < k \; \forall \, e \in E(D)$

(2) $f^+(v) \equiv f^-(v) (\text{mod } k)$

Let $D(f) \coloneqq \sum_{v \in V(D)} |f^+(v) - f^-(v)|$, and choose f satisfying (1) and (2) with $D(f)$ minimum. WMA $f(e) > 0 \; \forall \, e \in E(D)$. Let

$$A = \{v \in V(D): f^+(v) > f^-(v)\}$$

and

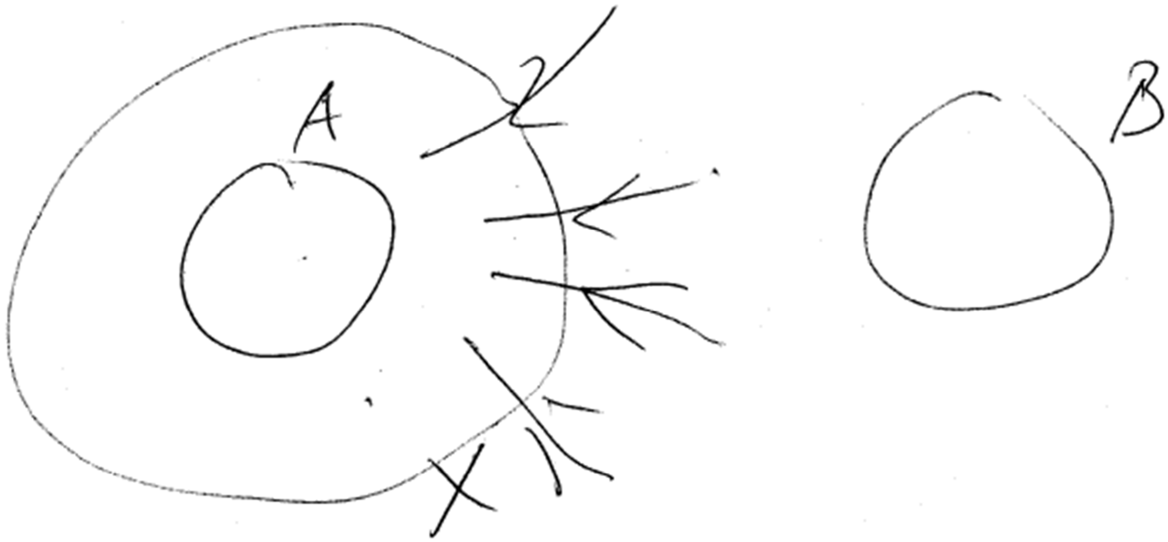
$$B = \{v \in V(D): f^+(v) < f^-(v)\}$$

Claim. $\nexists$ directed $A \to B$ path.

Pf. O.w. decrease the flow by k along such path. If $A = B = \emptyset$, then done, so WMA one is empty, and hence both are, because

$$\sum_{v \in V(D)} f^+(v) = \sum_{e \in E(D)} f(e) = \sum_{v \in V(D)} f^-(v)$$

By the claim $\exists \, X$ with $A \subseteq X$, $B \cap X = \emptyset$ and $\delta^+(X) = \emptyset$.



$$\sum_{v\in X} f^+(v) > \sum_{v\in X} f^-(v)$$

$$-\sum_{e\in\langle X\rangle} f(e) + \sum_{v\in X} f^+(v) > -\sum_{e\in\langle X\rangle} f(e) + \sum_{v\in X} f^-(v)$$

$0 = f^+(X) > f^-(X)$, a contradiction. □

Corollary. For a graph G and a group Γ, the following are equivalent:

(1) G is NZ Γ-flow

(2) G has a NZ k-flow, where $k = |\Gamma|$.

Corollary. A cubic graph has a NZ 4-flow if and only if it is 3-edge-colorable.

Proof. NZ 4-flow ⇔ NZ $Z_2 \times Z_2$-flow ⇔ 3-edge-coloring using the colors $(0,1), (1,0), (1,1)$

Thm. If G is plane, then G has a NZ k-flow if and only if G^* is k-colorable.

Corollary. The 4CT is equivalent to: Every 2-edge-connected cubic planar graph is 3-edge-colorable.

Thm. A cubic graph has a NZ 3-flow ⇔ it is bipartite.

Pf. NZ 3-flow ⇔$\mathbb{Z}_3$-flow ⇔∃ orientation s.t. $f = 1$ is a $\mathbb{Z}_3$-flow. Since G cubic ⇒ sources vs. sinks is a bipartition. That proves ⇒.

⇐ Direct one way ⇒$\mathbb{Z}_3$-flow □

3-flow conjecture. Every 4-edge-connected graph has a NZ 3-flow.

3-edge-coloring conjecture. Every 2-edge-connected cubic graph with no Petersen minor is 3-edge-colorable (⇔ NZ 4-flow).

4-flow conjecture. Every 2-edge-connected graph with no Petersen minor has a NZ 4-flow.

This implies

Grőtzsch conjecture. Let G be a planar graph of max degree 3 with no subgraph H s.t. H has all vertices of degree 3, except for exactly one of degree 2. Then G is 3-edge-colorable.

Implies the 4CT.

5-flow conjecture. Every 2-edge-connected graph has a NZ 5-flow.

Corollary. For a graph G and a group Γ, the following are equivalent:

(1) G is NZ Γ-flow

(2) G has a NZ k-flow, where $k = |\Gamma|$.

Corollary. A cubic graph has a NZ 4-flow if and only if it is 3-edge-colorable.

Proof. NZ 4-flow ⇔ NZ $Z_2 \times Z_2$-flow ⇔ 3-edge-coloring using the colors $(0,1), (1,0), (1,1)$

Thm. If G is plane, then G has a NZ k-flow if and only if G^* is k-colorable.

Corollary. The 4CT is equivalent to: Every 2-edge-connected cubic planar graph is 3-edge-colorable.

Application to Algebra

A ***ring*** is $(R, +, \cdot, 0, 1)$, where

- $(R, +)$ is an abelian group with identity 0
- $a(bc) = (ab)c$ and 1 is a multiplicative identity
- $(a+b)c = ac + bc$ and $c(a+b) = ca + cb$

Examples. (i) Z

(ii) Matrices over any ring

The ***commutator*** in a ring is $[a, b] = ab - ba$. More generally,

$$[a_1, a_2, \dots, a_k] = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)}$$

where the summation is over all permutations σ of $\{1,2,\dots,k\}$.

If $[a_1, a_2, \dots, a_k] = 0$ for all $a_i \in R$, then R is said to satisfy the $\boldsymbol{k^{th}}$ ***polynomial identity***.

THEOREM (Amitsur, Levitzky) The ring of $k \times k$ matrices over a commutative ring R satisfies the $(2k)^{\text{th}}$ polynomial identity. In other words, if $A_1, A_2, \dots, A_{2k}$ are $k \times k$ matrices with entries in R, then $[A_1, A_2, \dots, A_{2k}] = 0$.

Corollary. For a graph G and a group Γ, the following are equivalent:

(1) G is NZ Γ-flow

(2) G has a NZ k-flow, where $k = |\Gamma|$.

Corollary. A cubic graph has a NZ 4-flow if and only if it is 3-edge-colorable.

Proof. NZ 4-flow ⇔ NZ $Z_2 \times Z_2$-flow ⇔ 3-edge-coloring using the colors $(0,1), (1,0), (1,1)$

Thm. If G is plane, then G has a NZ k-flow if and only if G^* is k-colorable.

Corollary. The 4CT is equivalent to: Every 2-edge-connected cubic planar graph is 3-edge-colorable.

Application to Algebra

A ***ring*** is $(R, +, \cdot, 0, 1)$, where

- $(R, +)$ is an abelian group with identity 0
- $a(bc) = (ab)c$ and 1 is a multiplicative identity
- $(a + b)c = ac + bc$ and $c(a + b) = ca + cb$

Examples. (i) Z

(ii) Matrices over any ring

The ***commutator*** in a ring is $[a, b] = ab - ba$. More generally,

$$[a_1, a_2, \dots, a_k] = \sum_{\sigma} \operatorname{sgn}(\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(k)}$$

where the summation is over all permutations σ of $\{1, 2, \dots, k\}$.

If $[a_1, a_2, \dots, a_k] = 0$ for all $a_i \in R$, then R is said to satisfy the $\boldsymbol{k^{th}}$ ***polynomial identity***.

THEOREM (Amitsur, Levitzky) The ring of $k \times k$ matrices over a commutative ring R satisfies the $(2k)^{\text{th}}$ polynomial identity. In other words, if $A_1, A_2, \dots, A_{2k}$ are $k \times k$ matrices with entries in R, then $[A_1, A_2, \dots, A_{2k}] = 0$.

Proof. Since $[A_1, A_2, \dots, A_{2k}]$ is linear in each variable, it suffices to prove the theorem for matrices of the form E_{ij} (each entry 0, except $e_{ij} = 1$). So we must show

$$[E_{i_1 j_1}, E_{i_2 j_2}, \dots, E_{i_{2k} j_{2k}}] = 0.$$

Define a directed multigraph D by $V(D) = \{1,2, \dots, k\}$ and $E(D) = \{e_1, e_2, \dots, e_{2k}\}$, where $e_t = i_t j_t$. Then

$$E_{\sigma(e_1)} E_{\sigma(e_2)} \cdots E_{\sigma(e_{2k})} \neq 0$$

if and only if $\sigma(e_1), \sigma(e_2), \dots, \sigma(e_{2k})$ is the edge-set of an Euler trail in D. So

$$\left[E_{i_1 j_1}, E_{i_2 j_2}, \dots, E_{i_{2k} j_{2k}}\right] = \left[E_{e_1}, E_{e_2}, \dots, E_{e_{2k}}\right] =$$

$$\sum_{\sigma} \operatorname{sgn}(\sigma) E_{\sigma(e_1)} E_{\sigma(e_2)} \cdots E_{\sigma(e_{2k})} = \sum_{W} \operatorname{sgn}(W) E_{xy}$$

where the last summation is over all Euler trails, $\operatorname{sgn}(W)$ is the sign of the permutation of $E(D)$ determined by W, x is the origin of W and y is its terminus. Furthermore,

$$\sum_{W} \operatorname{sgn}(W) E_{xy} = \sum_{x,y \in V(D)} \left(\sum_{W} \operatorname{sgn}(W) \right) E_{xy}$$

where the last summation is over all Euler trails from x to y.

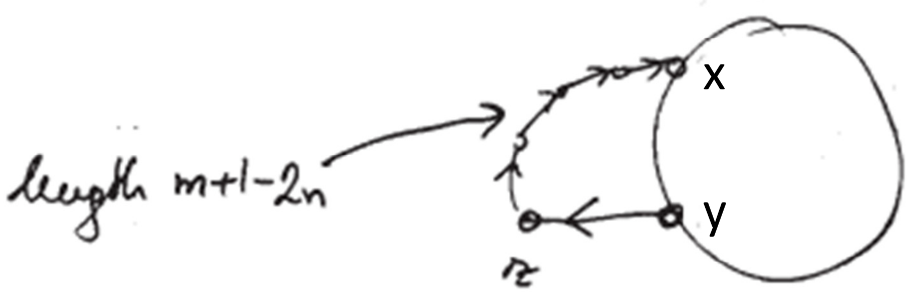
Lemma. Let D be a directed multigraph with $|E(D)| \geq 2|V(D)|$ and the let $x, y \in V(D)$. Then

$$\varepsilon(D,x,y) := \sum_{W} \mathrm{sgn}(W) = 0$$

where the summation is over all Euler trails from x to y.

Proof. Let $n = |V(D)|$ and $m = |E(D)|$. WMA no isolated vertices.

Step 1. We show that it suffices to prove the theorem for $x = y$ and $m = 2n$. We construct a directed multigraph D' by adding a new vertex z, an edge yz and a path $z \to x$ of length $m + 1 - 2n$.



Then $|V(D')| = n + m + 1 - 2n = m + 1 - n$

$$|E(D')| = m + m + 1 - 2n + 1 = 2(m + 1 - n)$$

$$|\varepsilon(D,x,y)| = |\varepsilon(D',z,z)|$$

Proof. Since $[A_1, A_2, \ldots, A_{2k}]$ is linear in each variable, it suffices to prove the theorem for matrices of the form E_{ij} (each entry 0, except $e_{ij} = 1$). So we must show

$$[E_{i_1 j_1}, E_{i_2 j_2}, \ldots, E_{i_{2k} j_{2k}}] = 0.$$

Define a directed multigraph D by $V(D) = \{1,2, \ldots, k\}$ and $E(D) = \{e_1, e_2, \ldots, e_{2k}\}$, where $e_t = i_t j_t$. Then

$$E_{\sigma(e_1)} E_{\sigma(e_2)} \cdots E_{\sigma(e_{2k})} \neq 0$$

if and only if $\sigma(e_1), \sigma(e_2), \ldots, \sigma(e_{2k})$ is the edge-set of an Euler trail in D. So

$$\left[E_{i_1 j_1}, E_{i_2 j_2}, \ldots, E_{i_{2k} j_{2k}}\right] = \left[E_{e_1}, E_{e_2}, \ldots, E_{e_{2k}}\right] =$$

$$\sum_{\sigma} \operatorname{sgn}(\sigma) E_{\sigma(e_1)} E_{\sigma(e_2)} \cdots E_{\sigma(e_{2k})} = \sum_{W} \operatorname{sgn}(W) E_{xy}$$

where the last summation is over all Euler trails, $\operatorname{sgn}(W)$ is the sign of the permutation of $E(D)$ determined by W, x is the origin of W and y is its terminus. Furthermore,

$$\sum_{W} \operatorname{sgn}(W) E_{xy} = \sum_{x,y \in V(D)} \left(\sum_{W} \operatorname{sgn}(W) \right) E_{xy}$$

where the last summation is over all Euler trails from x to y.

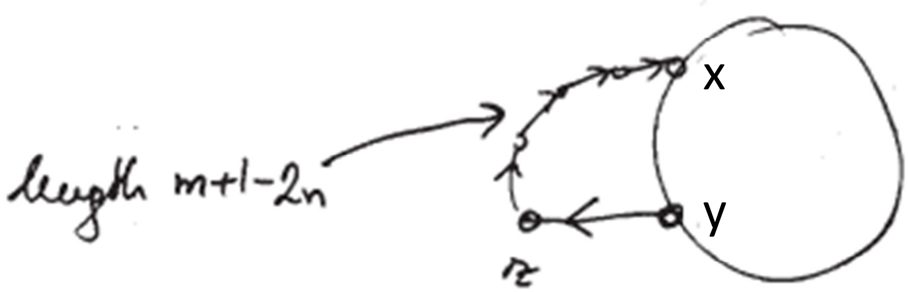
Lemma. Let D be a directed multigraph with $|E(D)| \geq 2|V(D)|$ and the let $x, y \in V(D)$. Then

$$\varepsilon(D, x, y) := \sum_{W} \operatorname{sgn}(W) = 0$$

where the summation is over all Euler trails from x to y.

Proof. Let $n = |V(D)|$ and $m = |E(D)|$. WMA no isolated vertices.

Step 1. We show that it suffices to prove the theorem for $x = y$ and $m = 2n$. We construct a directed multigraph D' by adding a new vertex z, an edge yz and a path $z \to x$ of length $m + 1 - 2n$.



Then $|V(D')| = n + m + 1 - 2n = m + 1 - n$

$$|E(D')| = m + m + 1 - 2n + 1 = 2(m + 1 - n)$$

$$|\varepsilon(D, x, y)| = |\varepsilon(D', z, z)|$$

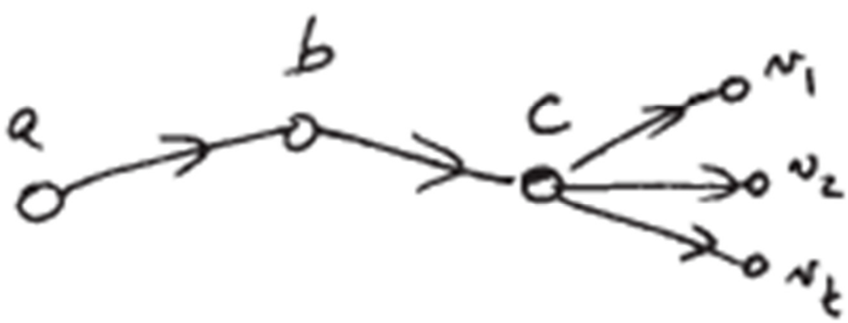
We proceed by induction on n. WMA D has a closed Euler trail, for otherwise $\varepsilon(D, z, z) = 0$. Thus $\deg^+(v) = \deg^-(v)$ for all $v \in V(D)$.

Step 2. If D has a parallel edge, then $\varepsilon(D, z, z) = 0$.

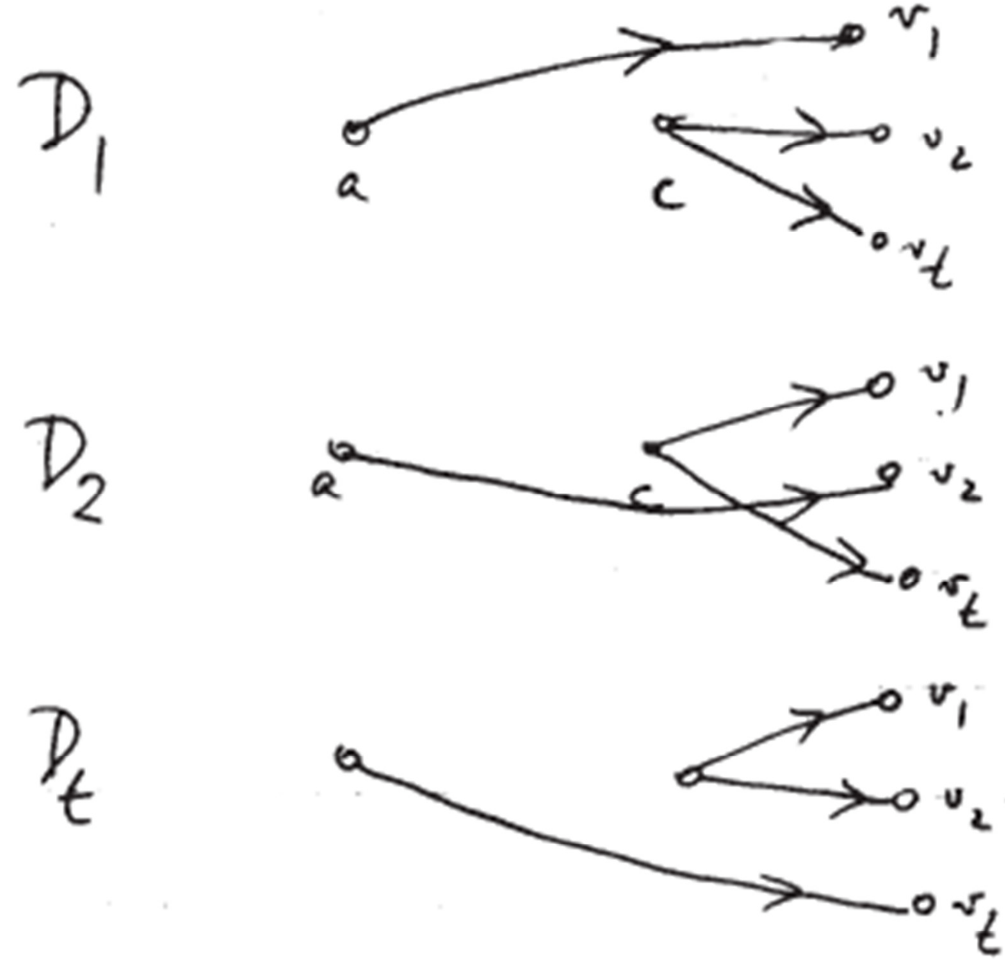
Step 3. If D has a vertex $b \neq z$ of degree 2 with one neighbor

Delete b and go by induction

Step 4. If D has a vertex $b \neq z$ of degree 2 with two neighbors



Define



Then

$$|\varepsilon(D,z,z)| = \left|\sum_{i=1}^{t} \varepsilon(D_i,z,z)\right|$$

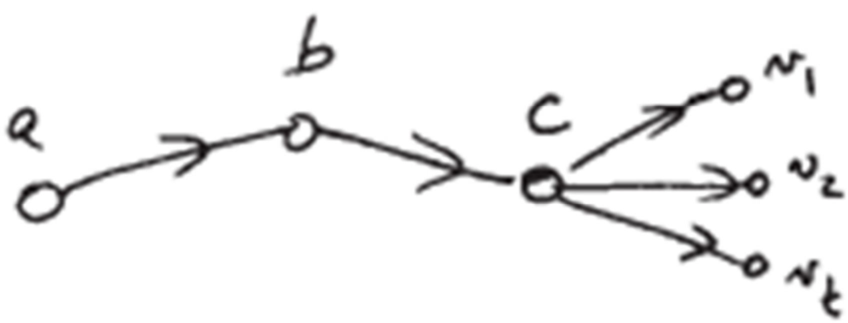
We proceed by induction on n. WMA D has a closed Euler trail, for otherwise $\varepsilon(D, z, z) = 0$. Thus $\deg^+(v) = \deg^-(v)$ for all $v \in V(D)$.

Step 2. If D has a parallel edge, then $\varepsilon(D, z, z) = 0$.

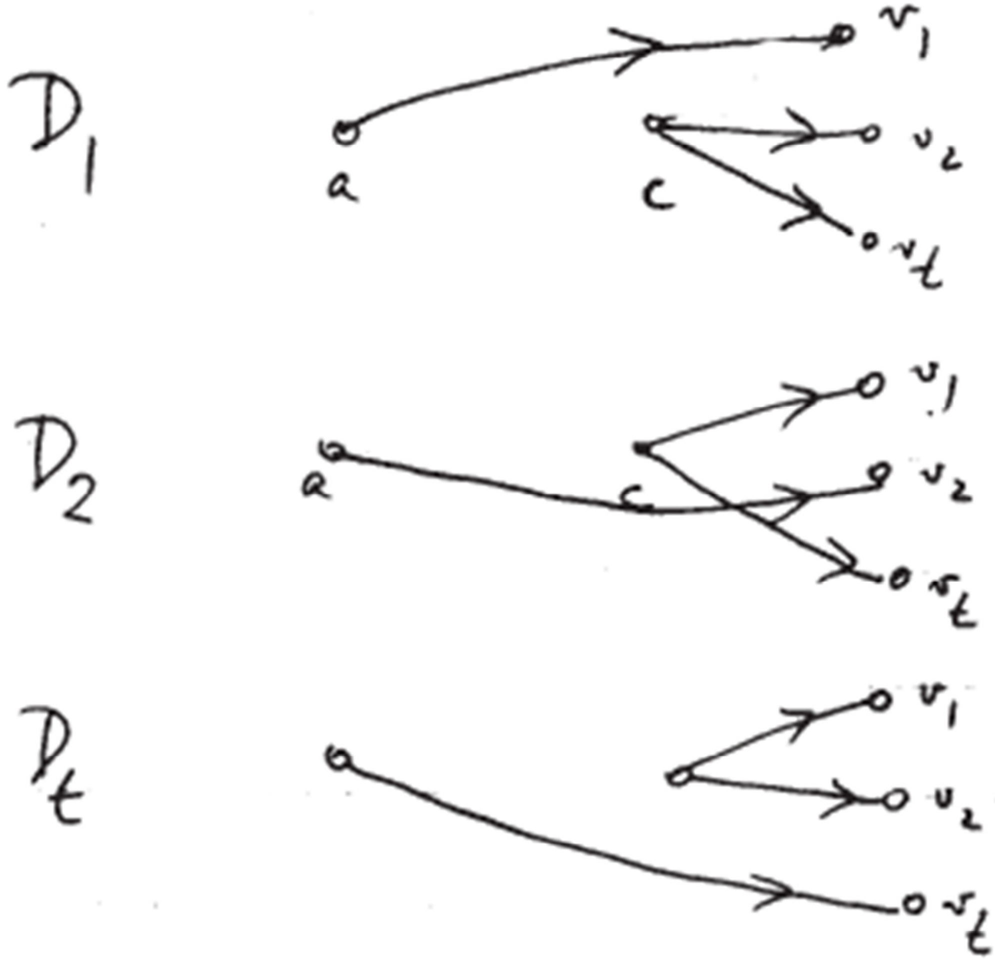
Step 3. If D has a vertex $b \neq z$ of degree 2 with one neighbor

Delete b and go by induction

Step 4. If D has a vertex $b \neq z$ of degree 2 with two neighbors



Define



Then

$$|\varepsilon(D,z,z)| = \left|\sum_{i=1}^{t} \varepsilon(D_i,z,z)\right|$$

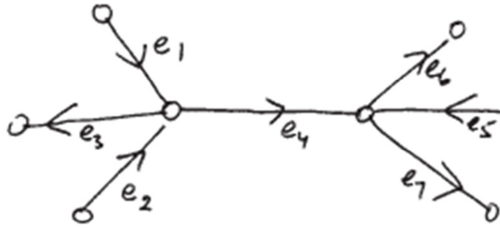
Step 5. If D has a loop at a vertex $b \neq z$ of degree 4

$$|\varepsilon(D, z, z)| = |\varepsilon(D', z, z)|$$

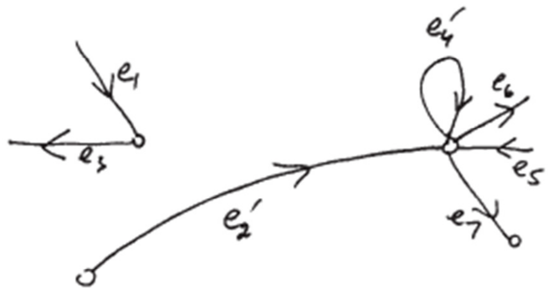
Step 6. If none of the above apply, then either

- $\deg^+(v) = \deg^-(v) = 2$ for all $v \in V(D)$, or
- $\deg^+(z) = 1$, $\deg^+(w) = 3$, $\deg^+(v) = 2$ for other v.

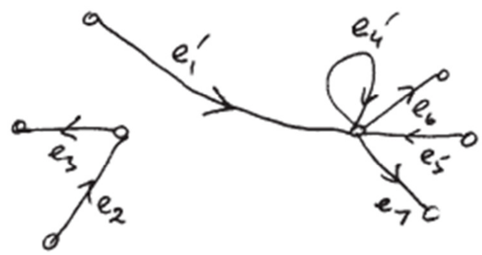
Step 7. There exist two adjacent vertices of outdegree two (exercise)



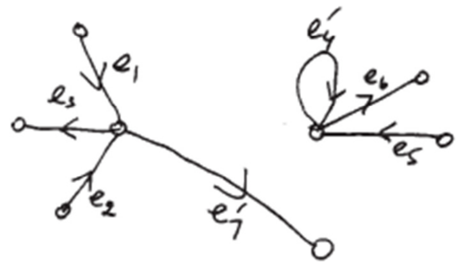
D_2



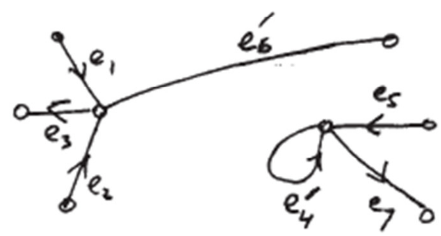
D_1



D_7



D_6



$$\varepsilon(D,z,z) = \varepsilon(D_1,z,z) + \varepsilon(D_2,z,z) - \varepsilon(D_6,z,z) - \varepsilon(D_7,z,z)$$

Question. Is the bound $|E(D)| \geq 2|V(D)|$ in the lemma best possible?

Step 5. If D has a loop at a vertex $b \neq z$ of degree 4

$$|\varepsilon(D, z, z)| = |\varepsilon(D', z, z)|$$

Step 6. If none of the above apply, then either

- $\deg^+(v) = \deg^-(v) = 2$ for all $v \in V(D)$, or
- $\deg^+(z) = 1,\ \deg^+(w) = 3, \deg^+(v) = 2$ for other v.

Step 7. There exist two adjacent vertices of outdegree two (exercise)