What is $\det(M+E_{vv})$? WMA $v=1$.

$$\det(M+E_{11})=\sum_{\sigma}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}(M+E_{11})_{i\sigma(i)}=$$

$$=\sum_{\sigma:\sigma(1)=1}\operatorname{sgn}(\sigma)\,(M_{11}+1)\prod_{i=2}^{n}M_{i\sigma(i)}+$$

$$+\sum_{\sigma:\sigma(1)\neq 1}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}M_{i\sigma(i)}=$$

$$=\sum_{\sigma}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}M_{i\sigma(i)}+\sum_{\sigma:\sigma(i)=1}\operatorname{sgn}(\sigma)\prod_{i=2}^{n}M_{i\sigma(i)}=$$

$$=\det(M)+\det M(1)$$

Proposition. Let G be a multigraph, and let e be an edge that is not a loop. Then

$$\tau(G) = \tau(G \backslash e) + \tau(G/e)$$

Definition. Let G be a multigraph with $V(G) = \{1,2, \dots, n\}$ and let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix defined by

$$a_{ij} = \#\text{ edges with ends } i, j$$

Then A is called the ***adjacency matrix*** of G.

The ***Laplacian matrix*** of a graph is defined by $L = (\ell_{ij})$ where

$$\ell_{ij} = \begin{cases} \sum_{k \neq i} a_{ik} & \text{if } i = j \\ -a_{ij} & \text{otherwise} \end{cases}$$

Note that rows and columns sum to 0, and hence $\det L = 0$.

Kirchhoff's Matrix Tree Theorem. Let G be a multigraph, let L be its Laplacian matrix, let $k \in \{1,2, \dots, k\}$, and let $L(k)$ denote the matrix obtained from L by deleting the k^{th} row and k^{th} column. Then $\tau(G) = \det L(k)$.

Proof. If G is disconnected, then $\tau(G) = 0 = \det L(k)$.

WMA G is connected and loopless.

If $|E(G)| = 0$, then $\tau(G) = 1 = \det L(k)$.

We proceed by induction on $|E(G)|$.

Recall

$$\tau(G) = \tau(G \backslash e) + \tau(G/e)$$

Enough to show

$$(*) \qquad \det L_G(u) = \det L_{G\backslash e}(u) + \det L_{G/e}(w)$$

where $e = uv$ and w is the new vertex of G/e

$$L_G(u) = L_{G\backslash e}(u) + E_{vv}$$

$$\begin{aligned} \det L_G(u) &= \det[L_{G\backslash e}(u) + E_{vv}] \\ &= \det L_{G\backslash e}(u) + \det L_{G\backslash e}(u,v) \\ &= \det L_{G\backslash e}(u) + \det L_{G/e}(w) \end{aligned}$$

This proves $(*)$, and hence the theorem. $\square$

Theorem. (Cayley) $\tau(K_n) = n^{n-2}$. In other words there are exactly n^{n-2} trees with vertex-set $\{1,2,\dots,n\}$.

Proof. By Kirchhoff's Matrix Tree Theorem

$$\tau(K_n) = \det\left.\begin{pmatrix} n-1 & -1 & & -1 \\ -1 & n-1 & & \\ & & & -1 \\ -1 & & -1 & n-1 \end{pmatrix}\right\} n-1 =$$

[by adding rows $2,3,\dots,n-1$ to row 1]

$$= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & n-1 & -1 & -1 \\ \vdots & & & \\ -1 & & -1 & n-1 \end{pmatrix} =$$

[by adding row 1 to all other rows]

$$= \det\begin{pmatrix} 1 & 1 & & 1 \\ & n & & \\ & & & 0 \\ 0 & & & n \end{pmatrix} = n^{n-2}$$

□

A ***directed multigraph*** is a triple (V, E, ψ), where V, E are finite sets and ψ is an incidence relation that assigns to every edge $e \in E$ an ordered pair of not necessarily distinct vertices of V, called its ***ends***.

We denote the outdegree of a vertex v by $\deg^+(v)$ and the indegree by $\deg^-(v)$.

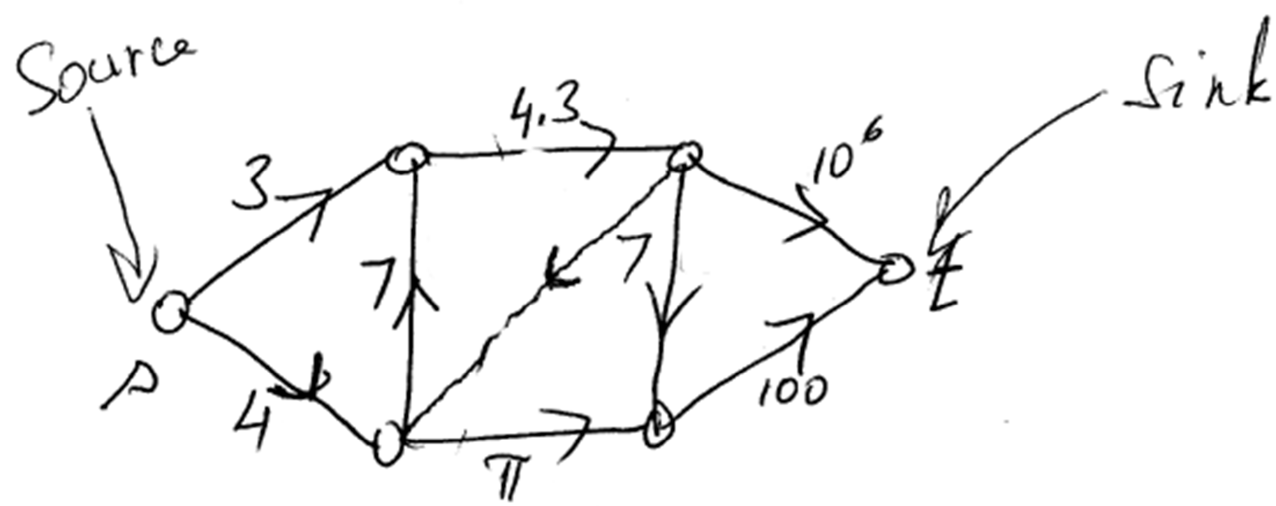
The max-flow min-cut theorem

A ***directed graph*** or ***digraph*** is a pair $D = (V, E)$, where V is a finite set and $E \subseteq V \times V$.

A ***network*** is a quadruple $N = (D, s, t, c)$, where D is a digraph, $s, t \in V(D)$ are distinct, and $c: E(D) \to [0, \infty]$.

s … source   t … sink   c … capacity function

Example



Notation. For $X \subseteq V(D)$

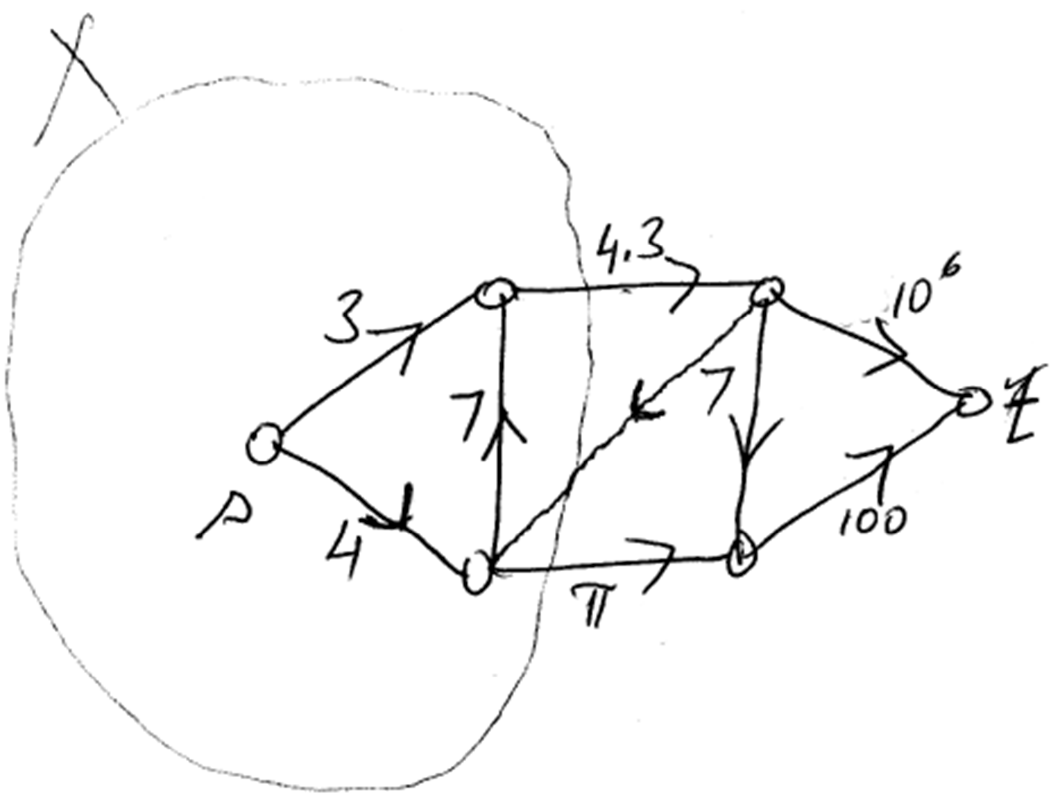
$$\delta^+(X) := \{e \in E(D) : e \text{ has tail in } X, \text{head in } V(D) - X\}$$

$$\delta^-(X) := \{e \in E(D) : e \text{ has head in } X, \text{tail in } V(D) - X\}$$

$\delta^+(\{v\}) = \delta^+(v)$, $\delta^-(\{v\}) = \delta^-(v)$. If $f : E(D) \to \mathbb{R}$, then

$$f^+(X) := \sum_{e \in \delta^+(X)} f(e), \quad f^-(X) := \sum_{e \in \delta^-(X)} f(e)$$

Example

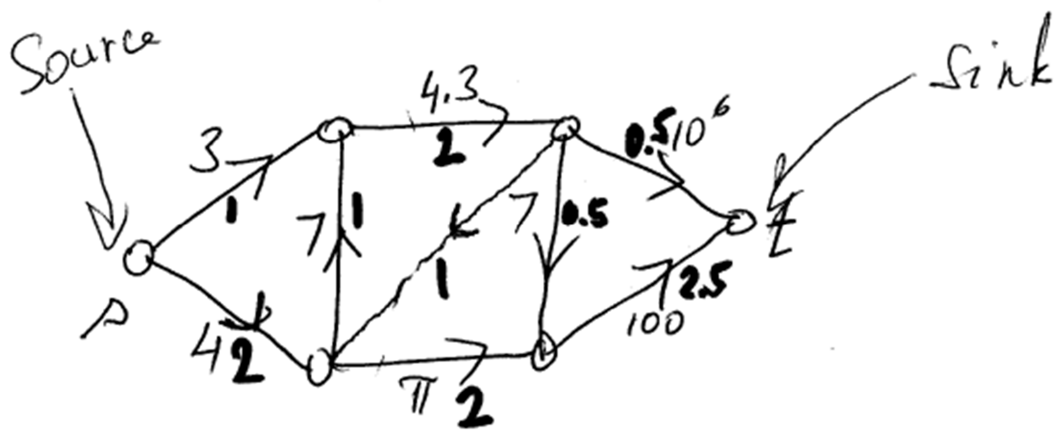


A ***flow*** in N is a mapping $f: E(D) \to R$ such that

(i) $0 \le f(e) \le c(e) \quad \forall e \in E(D)$ (capacity constraints)

(ii) $f^{+}(v) = f^{-}(v) \quad \forall\, v \in V(D) - \{s, t\}$
(conservation conditions)

Example



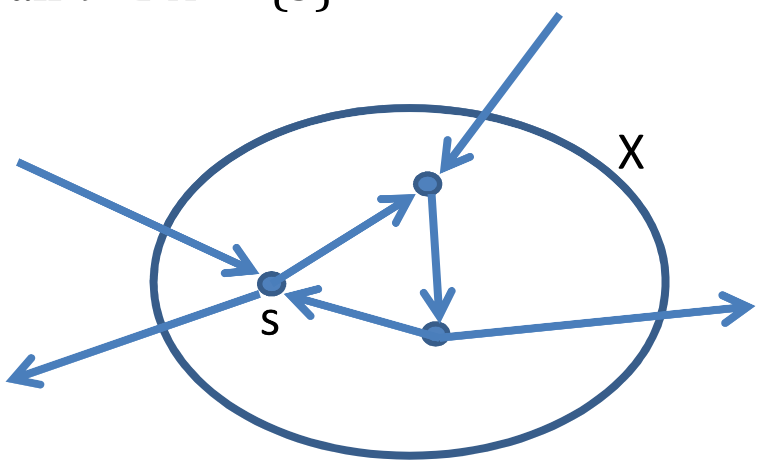
Lemma. If f is a flow in a network $N = (D, s, t, c)$ and $X \subseteq V(D)$ with $s \in X$, $t \notin X$, then

$$f^+(s) - f^-(s) = f^+(X) - f^-(X)$$

Definition. $f^+(s) - f^-(s)$ is the ***value*** of f, denoted by $\mathrm{val}(f)$.

Proof. $f^+(v) = f^-(v) \quad \forall v \in V(D) - \{s, t\}$.

Sum over all $v \in X - \{s\}$



$$f^+(X) - f(s, X^c) + f(X, s) + \sum_{\substack{e \text{ has both} \\ \text{ends in } X - \{s\}}} f(e) =$$

$$= f^-(X) - f(X^c, s) + f(s, X) + \sum_{\substack{e \text{ has both} \\ \text{ends in } X - \{s\}}} f(e)$$

Corollary. If f is a flow in a network $N = (D, s, t, c)$, then

$$f^+(s) - f^-(s) = f^-(t) - f^+(t)$$

Proof. Apply previous lemma to $X = V(D) - \{t\}$. □

A ***cut*** in a network N is a set of edges of the form $\delta^+(X)$ for some set $X \subseteq V(D)$ with $s \in X$, $t \notin X$. The ***capacity*** of a cut K is

$$\text{cap}\,(K) := \sum_{e \in K} c(e)$$

If $K = \delta^+(X)$, then $\text{cap}(K) = c^+(X)$.

Corollary. $\text{val}(f) \leq \text{cap}(K)$ for every flow f and every cut K in N.

Proof. Let $K = \delta^+(X)$.

$$\text{val}(f) = f^+(s) - f^-(s) = f^+(X) - f^-(X) =$$

$$= \sum_{e \in \delta^+(X)} f(e) - \sum_{e \in \delta^-(X)} f(e) \leq \sum_{e \in \delta^+(X)} c(e) = \text{cap}(K)$$

□