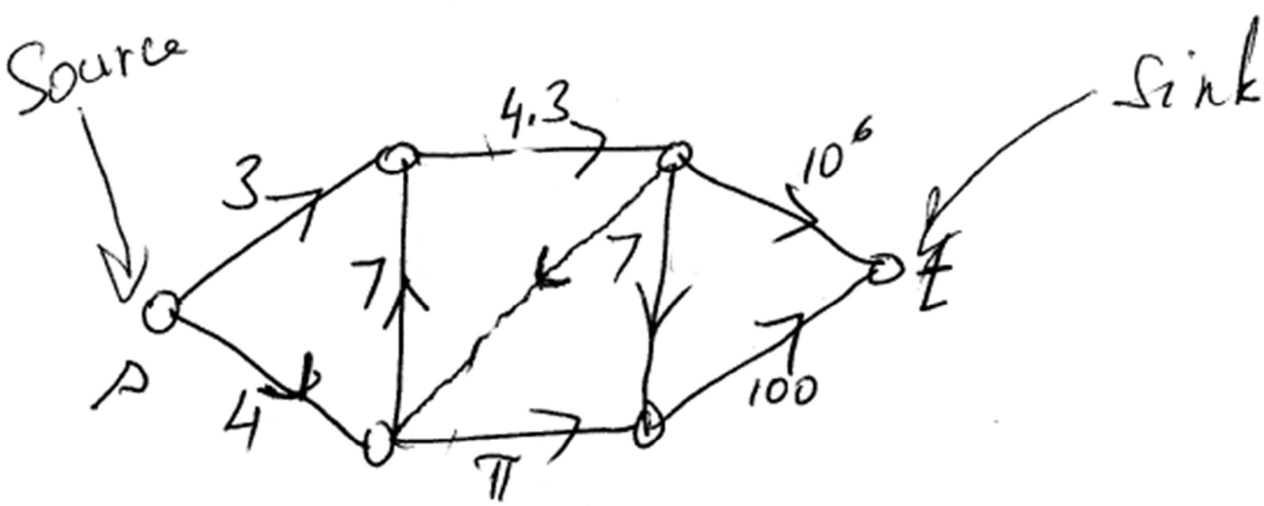
The max-flow min-cut theorem

A ***directed graph*** or ***digraph*** is a pair $D = (V, E)$, where V is a finite set and $E \subseteq V \times V$.

A ***network*** is a quadruple $N = (D, s, t, c)$, where D is a digraph, $s, t \in V(D)$ are distinct, and $c: E(D) \to [0, \infty]$.

s … source t … sink c … capacity function

Example



Notation. For $X \subseteq V(D)$

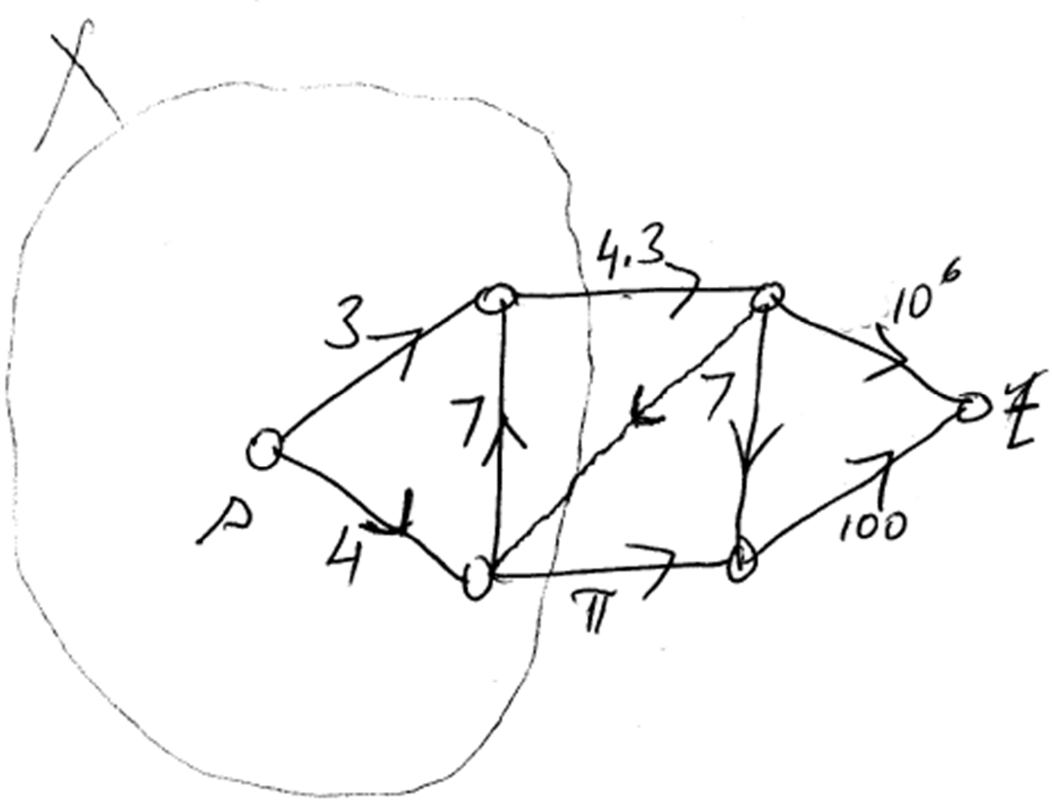
$$\delta^+(X) := \{e \in E(D) : e \text{ has tail in } X, \text{ head in } V(D) - X\}$$

$$\delta^-(X) := \{e \in E(D) : e \text{ has head in } X, \text{ tail in } V(D) - X\}$$

$\delta^+(\{v\}) = \delta^+(v)$, $\delta^-(\{v\}) = \delta^-(v)$. If $f : E(D) \to \mathbb{R}$, then

$$f^+(X) := \sum_{e \in \delta^+(X)} f(e), \qquad f^-(X) := \sum_{e \in \delta^-(X)} f(e)$$

Example

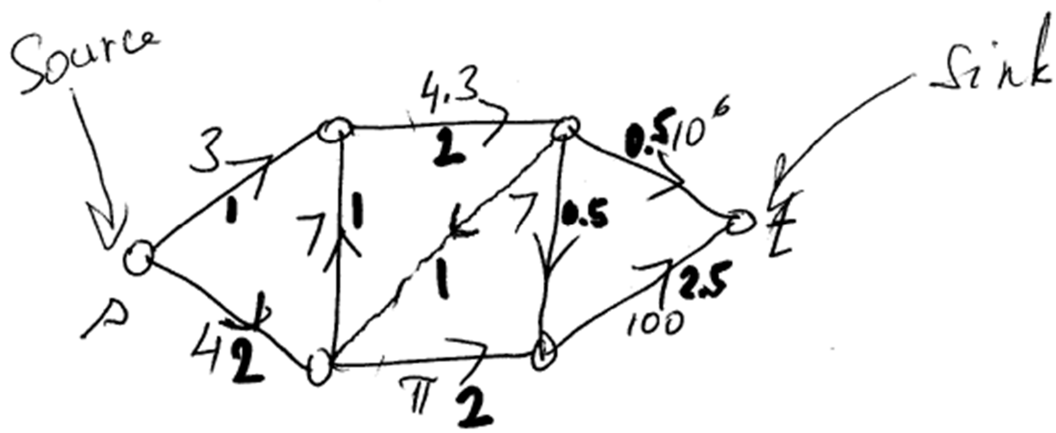


A ***flow*** in N is a mapping $f: E(D) \to R$ such that

(i) $0 \leq f(e) \leq c(e) \quad \forall e \in E(D)$ (capacity constraints)

(ii) $f^+(v) = f^-(v) \quad \forall\, v \in V(D) - \{s, t\}$
(conservation conditions)

Example



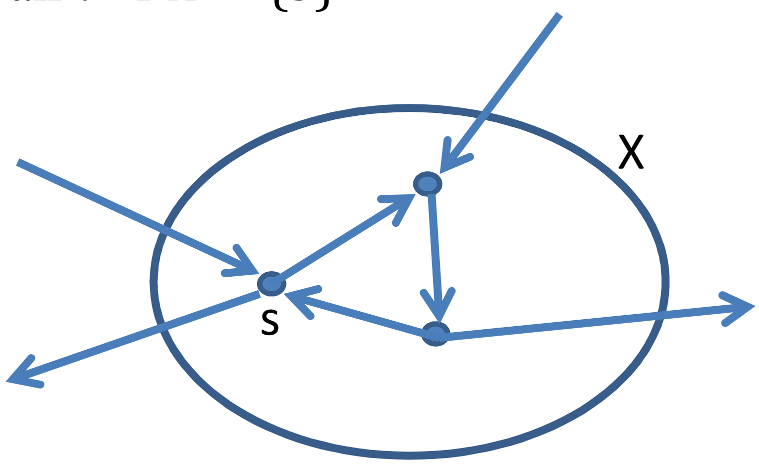
Lemma. If f is a flow in a network $N = (D, s, t, c)$ and $X \subseteq V(D)$ with $s \in X$, $t \notin X$, then

$$f^+(s) - f^-(s) = f^+(X) - f^-(X)$$

Definition. $f^+(s) - f^-(s)$ is the ***value*** of f, denoted by $\mathrm{val}(f)$.

Proof. $f^+(v) = f^-(v) \quad \forall v \in V(D) - \{s, t\}$.

Sum over all $v \in X - \{s\}$



$$f^+(X) - f(s, X^c) + f(X, s) + \sum_{\substack{e \text{ has both} \\ \text{ends in } X - \{s\}}} f(e) =$$

$$= f^-(X) - f(X^c, s) + f(s, X) + \sum_{\substack{e \text{ has both} \\ \text{ends in } X - \{s\}}} f(e)$$

$$f(s, X^c) + f(s, X) = f^+(s), \;\; f(X^c, s) + f(X, s) = f^-(s)$$

Corollary. If f is a flow in a network $N = (D, s, t, c)$, then

$$f^+(s) - f^-(s) = f^-(t) - f^+(t)$$

Proof. Apply previous lemma to $X = V(D) - \{t\}$. $\square$

A ***cut*** in a network N is a set of edges of the form $\delta^+(X)$ for some set $X \subseteq V(D)$ with $s \in X$, $t \notin X$. The ***capacity*** of a cut K is

$$\text{cap}\,(K) := \sum_{e \in K} c(e)$$

If $K = \delta^+(X)$, then $\text{cap}(K) = c^+(X)$.

Corollary. $\text{val}(f) \leq \text{cap}(K)$ for every flow f and every cut K in N.

Proof. Let $K = \delta^+(X)$.

$$\text{val}(f) = f^+(s) - f^-(s) = f^+(X) - f^-(X) =$$

$$= \sum_{e \in \delta^+(X)} f(e) - \sum_{e \in \delta^-(X)} f(e) \leq \sum_{e \in \delta^+(X)} c(e) = \text{cap}(K)$$

$\square$

Lemma. In any network, there exists a flow of maximum value.

Proof. Let $N = (D, s, t, c)$ be a network, and let

$$M \coloneqq \sup\ \{\mathrm{val}(f) : f \text{ is a flow in } N\}.$$

There exists a sequence of flows $f_1, f_2, ...$ such that

$$val(f_n) \geq M - \frac{1}{n}.$$

Let $e_1, e_2, \dots, e_m$ be the edges of D. The sequence $\{f_n(e_1)\}_{n=1}^{\infty}$ has a convergent subsequence $\{f_n(e_1)\}_{n\in A_1}$. Let $f(e_1)$ be the limit. The sequence $\{f_n(e_2)\}_{n\in A_1}$ has a convergent subsequence $\{f_n(e_2)\}_{n\in A_2}$. Let $f(e_2)$ be the limit. After m steps we construct a flow f. Then f is a flow of value M. □

Theorem. (Max-flow min-cut theorem, Ford & Fulkerson) In any network N there exists a flow f^* and cut K^* such that

$$\mathrm{val}(f^*) = \mathrm{cap}(K^*).$$

If the capacity function is integral (takes on integer values only), then f^* can be chosen integral.

Proof. Let $N = (D, s, t, c)$, and let f be a flow of maximum value. By an ***augmenting path*** we mean a path P in the underlying undirected multigraph such that

(i) s is an end of P

(ii) $f(e) < c(e)$ for every "forward" edge $e \in E(P)$

(iii) $f(e) > 0$ for every "backward" edge $e \in E(P)$