The max-flow min-cut theorem

A ***directed graph*** or ***digraph*** is a pair $D = (V, E)$, where V is a finite set and $E \subseteq V \times V$.

A ***network*** is a quadruple $N = (D, s, t, c)$, where D is a digraph, $s, t \in V(D)$ are distinct, and $c: E(D) \to [0, \infty]$.

s … source t … sink c … capacity function

Notation. For $X \subseteq V(D)$

$$\delta^+(X) := \{e \in E(D): e \text{ has tail in } X, \text{ head in } V(D) - X\}$$

$$\delta^-(X) := \{e \in E(D): e \text{ has head in } X, \text{ tail in } V(D) - X\}$$

$\delta^+(\{v\}) = \delta^+(v)$, $\delta^-(\{v\}) = \delta^-(v)$. If $f: E(D) \to \mathbb{R}$, then

$$f^+(X) := \sum_{e \in \delta^+(X)} f(e), \quad f^-(X) := \sum_{e \in \delta^-(X)} f(e)$$

A ***flow*** in N is a mapping $f: E(D) \to R$ such that

(i) $0 \leq f(e) \leq c(e) \quad \forall e \in E(D)$ (capacity constraints)

(ii) $f^+(v) = f^-(v) \quad \forall\, v \in V(D) - \{s, t\}$
(conservation conditions)

Lemma. If f is a flow in a network $N = (D, s, t, c)$ and $X \subseteq V(D)$ with $s \in X$, $t \notin X$, then

$$f^+(s) - f^-(s) = f^+(X) - f^-(X)$$

Definition. $f^+(s) - f^-(s)$ is the ***value*** of f, denoted by $\mathrm{val}(f)$.

Corollary. If f is a flow in a network $N = (D, s, t, c)$, then

$$f^+(s) - f^-(s) = f^-(t) - f^+(t)$$

A ***cut*** in a network N is a set of edges of the form $\delta^+(X)$ for some set $X \subseteq V(D)$ with $s \in X$, $t \notin X$. The ***capacity*** of a cut K is

$$\mathrm{cap}\,(K) \coloneqq \sum_{e \in K} c(e)$$

If $K = \delta^+(X)$, then $\mathrm{cap}(K) = c^+(X)$.

Corollary. $\mathrm{val}(f) \leq \mathrm{cap}(K)$ for every flow f and every cut K in N.

Lemma. In any network, there exists a flow of maximum value.

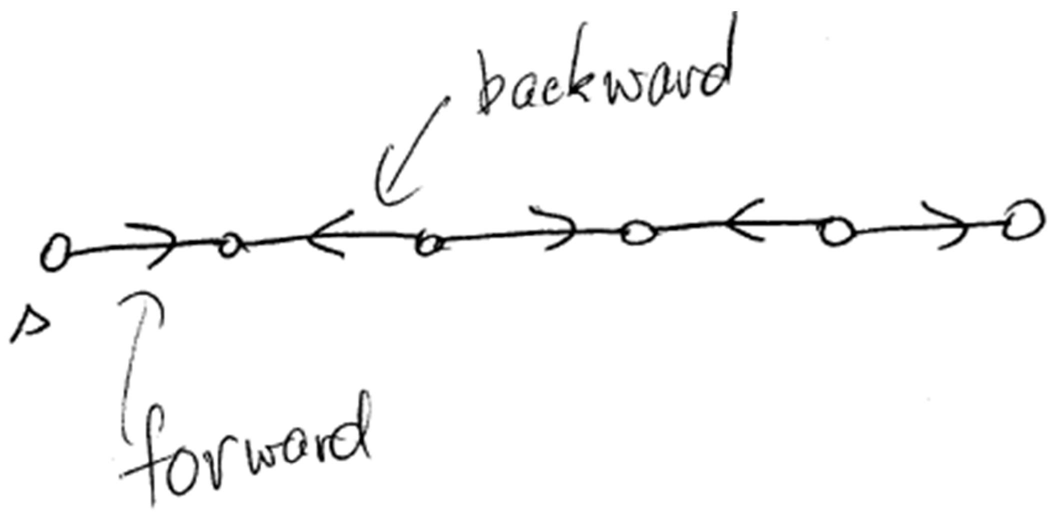
Theorem. (Max-flow min-cut theorem, Ford & Fulkerson) In any network N there exists a flow f^* and cut K^* such that

$$\mathrm{val}(f^*) = \mathrm{cap}(K^*).$$

If the capacity function is integral (takes on integer values only), then f^* can be chosen integral.

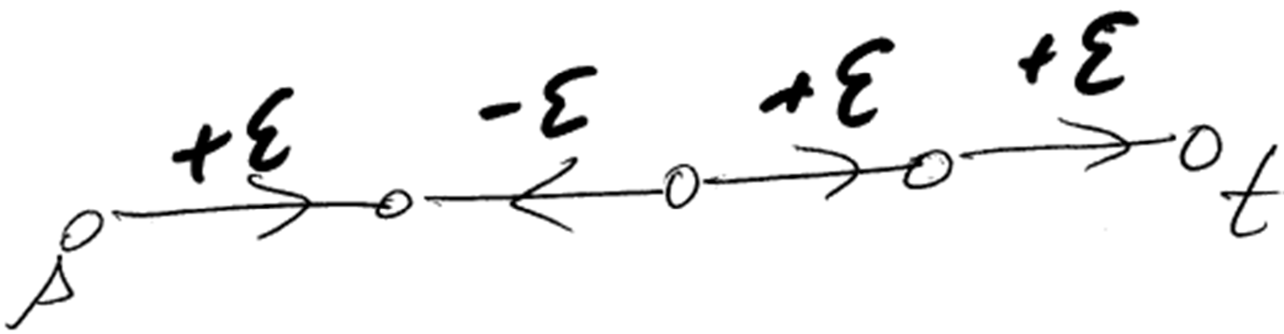
Proof. Let $N = (D, s, t, c)$, and let f be a flow of maximum value. By an ***augmenting path*** we mean a path P in the underlying undirected multigraph such that

(i) s is an end of P
(ii) $f(e) < c(e)$ for every "forward" edge $e \in E(P)$
(iii) $f(e) > 0$ for every "backward" edge $e \in E(P)$



Claim. There is no augmenting path from s to t.

Proof. Suppose for a contradiction that P is an augmenting path from s to t.



Let $\epsilon > 0$ be such that

$$f(e) + \epsilon \leq c(e) \quad \text{for every forward edge } e \in E(P)$$

$$f(e) \geq \epsilon \qquad \text{for every backward edge } e \in E(P)$$

Let f' be defined by

$$f'(e) = \begin{cases} f(e) & \text{if } e \notin E(P) \\ f(e) + \epsilon & \text{if } e \in E(P) \text{ is forward} \\ f(e) - \epsilon & \text{if } e \in E(P) \text{ is backward} \end{cases}$$

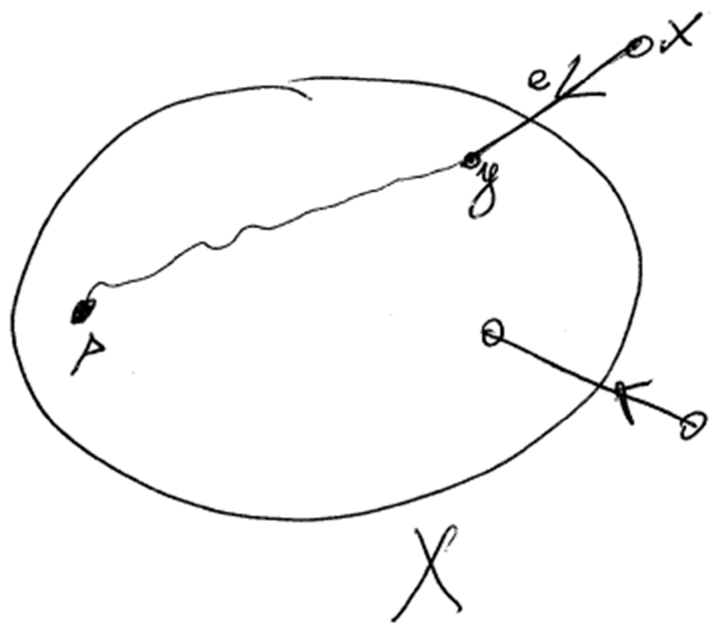
Then f' is a flow of value $\text{val}(f) + \epsilon > \text{val}(f)$, a contradiction. This proves the claim.

Define

$$X := \{v : \text{there is an augmenting path from } s \text{ to } v\}.$$

Then $t \notin X$. Let $K := \delta^+(X)$. Claim f, K are as desired.

We have $f(e) = 0$ for every edge $e = xy \in \delta^-(X)$. To see that let Q be an s-y augmenting path (which exists because $y \in X$). Then $Q + e$ is an s-x augmenting path, contrary to $x \notin X$. This shows $f(e) = 0$ for every edge $e \in \delta^-(X)$.



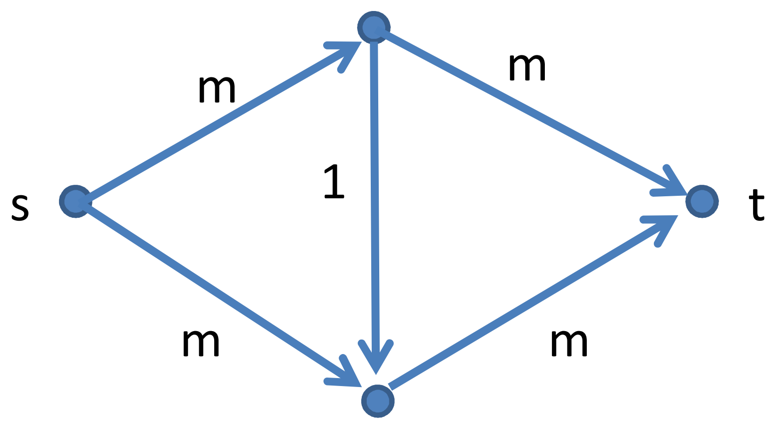
Similarly, $f(e) = c(e)$ for every edge $e \in \delta^+(X)$. Thus $f^+(X) = c^+(X)$ and $f^-(X) = 0$. Hence

$$\text{val}(f) = f^+(X) - f^-(X) = c^+(X) = \text{cap}(K).$$

If the capacity function is integral, then, starting from the zero flow, the proof constructs a maximal flow that is integral. □

The proof gives rise to an algorithm to construct a maximum flow and a minimum cut.

How good is the algorithm?



The algorithm may take $2m$ iterations

Size of input is about $s := \log_2 m$

So the running time is $2m = 2 \cdot 2^{\log_2 m} = 2 \cdot 2^{s}$

However, if at every step we pick a shortest augmenting path, then this leads to a polynomial-time algorithm (week 3 problem sets)

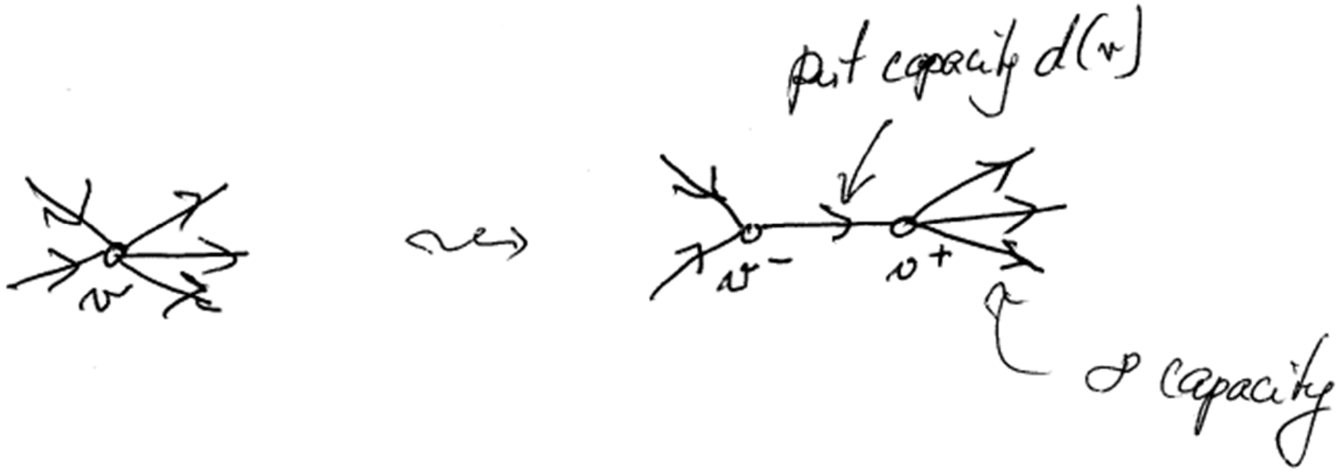
Multiple sources or sinks

Capacities on vertices

Suppose that $d: V(D) \to \mathbb{R}$ and we want our flow to satisfy

$$f^+(v) = f^-(v) \leq d(v) \text{ for every } v \in V(D) - \{s, t\}$$

An easy construction



Do this for every $v \in V(D) - \{s, t\}$.

Theorem. If we have capacities on vertices, then there exists a flow f^* and a *vertex-cut* K^* such that $\mathrm{val}(f^*) = \mathrm{cap}(K^*)$, where a vertex-cut is a set of vertices $K^* \subseteq V(D) - \{s, t\}$ such that there is no path from s to t in $D \backslash K^*$.

Proof. Use the above construction.

Application to baseball elimination

n teams; team i has w_i wins and r_{ij} games to play against team j. Team i is eliminated if it cannot finish with the most wins, or tied for the most wins.

Example.

	wins	to play	ATL	PHL	NY	MON
ATL	83	8		1	6	1
PHL	79	4	1		0	3
NY	78	7	6	0		1
MON	76	5	1	3	1	

MON is eliminated

ATL + NY have between them $\geq 83 + 78 + 6 = 167$ wins

On average they win 83.5 games ⇒ one of them wins 84 ⇒ PHL is eliminated.

Fix team i_0. Let

$$M := w_{i_0} + \sum_{j \neq i_0} r_{ij}$$

= max possible number of wins by team i_0.

Let A be a set of teams. If

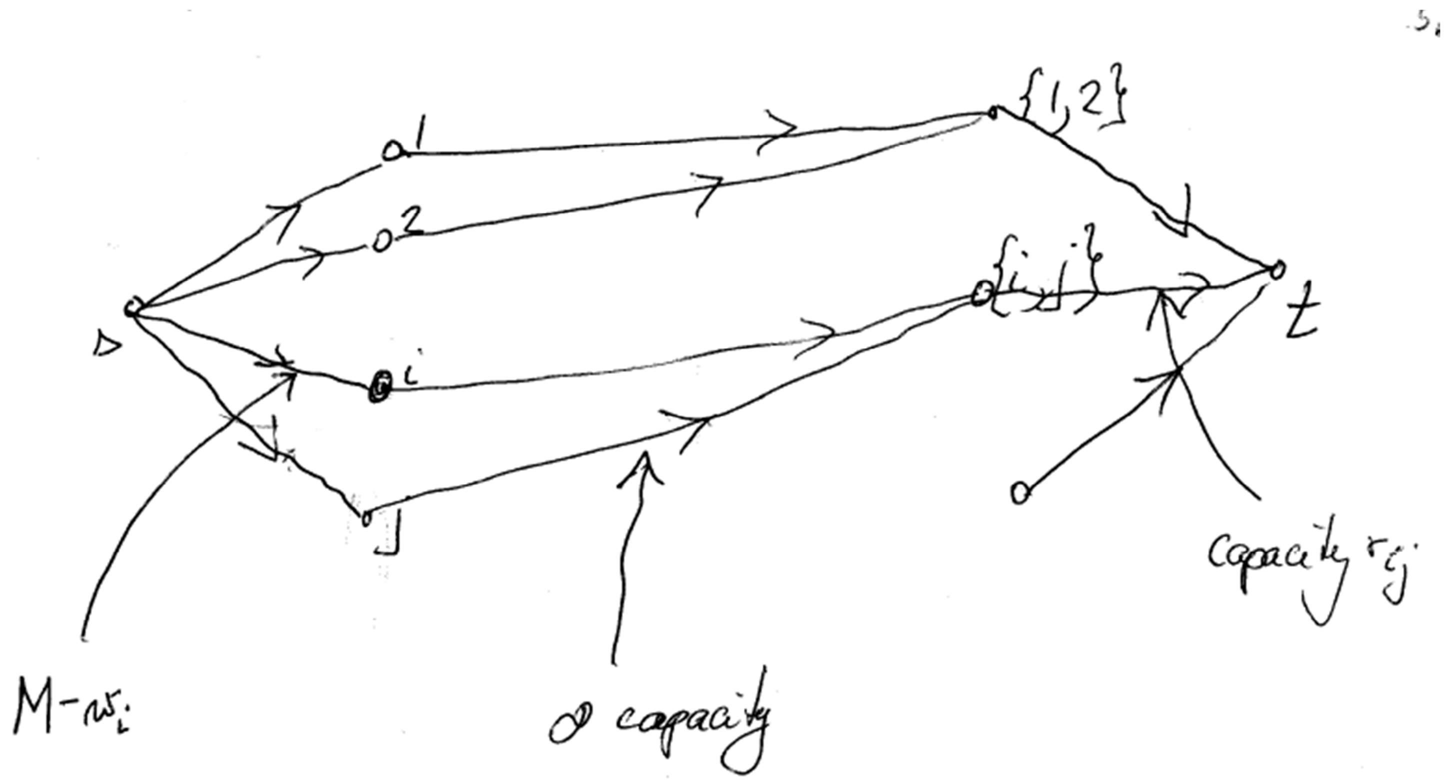
$$(*) \qquad \sum_{i \in A} w_i + \sum_{\substack{\{i,j\} \subseteq A \\ i \neq j}} r_{ij} > M|A|,$$

then some team in A will end up with $> M$ wins $\Rightarrow i_0$ is eliminated.

Theorem. Fix a team i_0. The team i_0 is eliminated if and only if there exists a set A satisfying $(*)$.

Proof. $\Leftarrow$ done above

$\Rightarrow$ construct a network using teams other than i_0



Case 1. $\exists$ flow of value $\sum_{\{i,j\}} r_{ij}$. Let y_{ij} be the flow on the edge $(i, \{i, j\})$. Then

$$y_{ij} + y_{ji} = r_{ij}$$

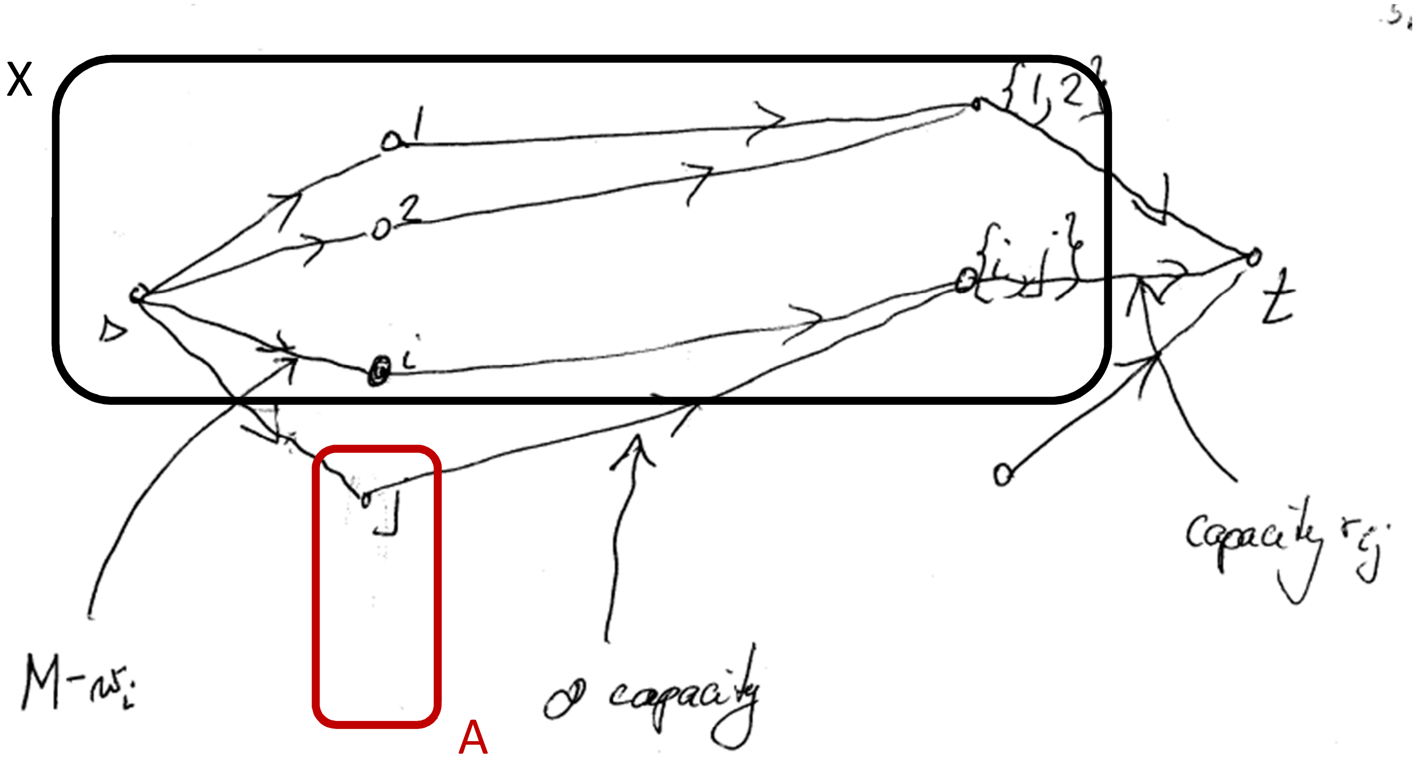
(flow conservation + edges into sink are used at capacity).

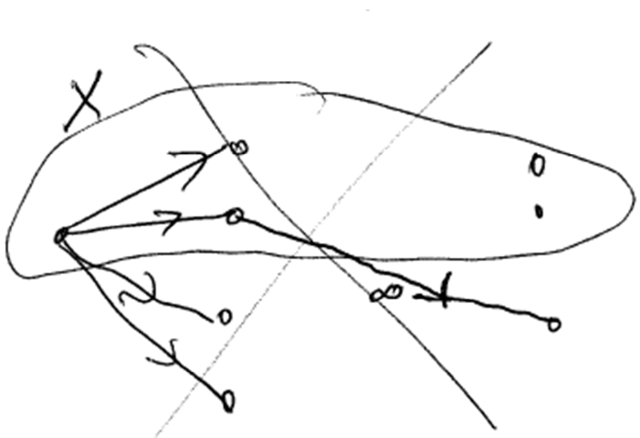
$$\sum_j y_{ij} \leq M - w_i \quad \forall\, i$$

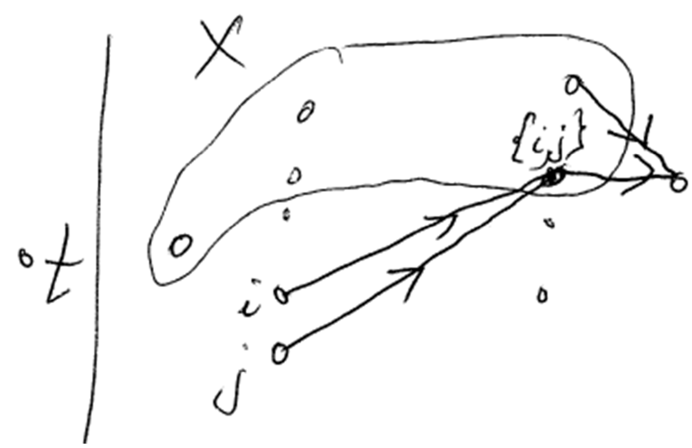
(flow conservation at i + capacity constraint on (s, i).

If team i wins y_{ij} games against team j, then team i ends up with $w_i + \sum_j y_{ij}$ wins $\leq M \Rightarrow i_0$ is not eliminated.

Case 2. $\nexists$ flow of value $\geq \sum_{\{i,j\}} r_{ij}$. By the MFMC theorem $\exists$ cut $\delta^+(X)$ of capacity $< \sum r_{ij}$. Let A be the set of teams not in X.







This cannot happen.

We may move $\{i, j\}$ out of X.

So the set of all $\{i,j\} \in X$ is equal to the set of all $\{i,j\}$ such that $\{i,j\} \nsubseteq A$.

The capacity of $\delta^+(X)$ is

$$\sum_{i \in A}(M - w_i) + \sum_{\{i,j\} \nsubseteq A} r_{ij} = M \cdot |A| - \sum_{i \in A} w_i + \sum_{\{i,j\} \nsubseteq A} r_{ij}.$$

This is $< \sum_{\{i,j\}} r_{ij}$, and so

$$M \cdot |A| - \sum_{i \in A} w_i < \sum_{\{i,j\} \subseteq A} r_{ij} \Rightarrow A \text{ satisfies } (*)$$

This proves that if team i_0 is eliminated, then there exists a set A satisfying $(*)$.