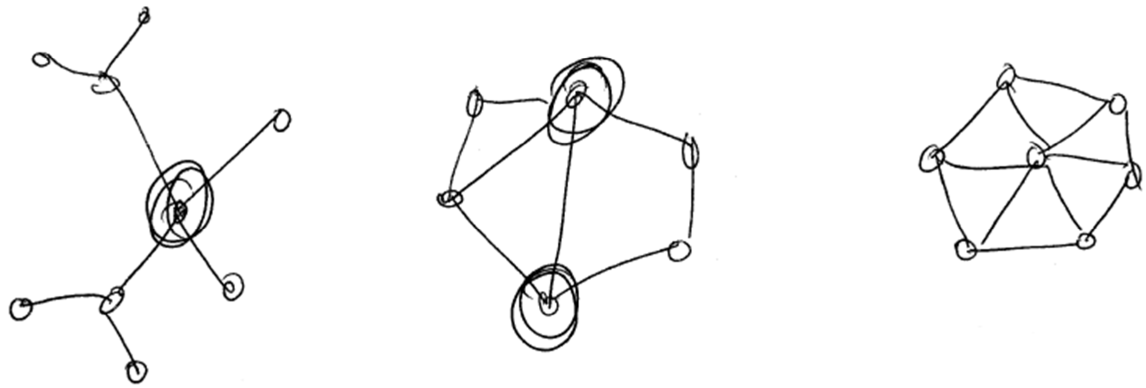
Connectivity



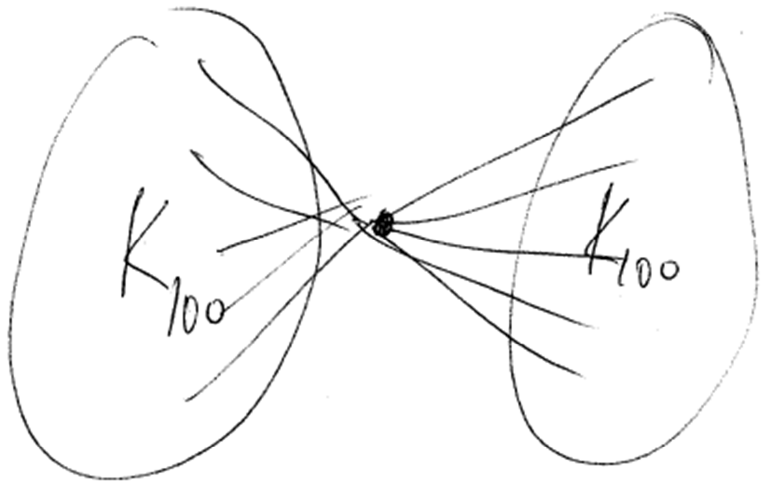
A ***vertex cut*** in G is a set $X \subseteq V(G)$ such that $G \backslash X$ is disconnected. We say that a graph G is k-***connected*** if it has $\geq k+1$ vertices and $G \backslash X$ is connected for every set $X \subseteq V(G)$ of size $< k$. We say that G is k-***edge-connected*** if $G \backslash F$ is connected for every set $F \subseteq E(G)$ of size $< k$.

The **connectivity** of G, denoted by $\kappa(G)$, is the maximum integer k such that G is k-connected.

The **edge-connectivity** of G, denoted by $\kappa'(G)$, is the maximum integer such that G is k-edge-connected.

Question. Is there a relationship between $\kappa(G)$ and $\kappa'(G)$?

There exist graphs with $\kappa(G) = 1$ and $\kappa'(G)$ big:

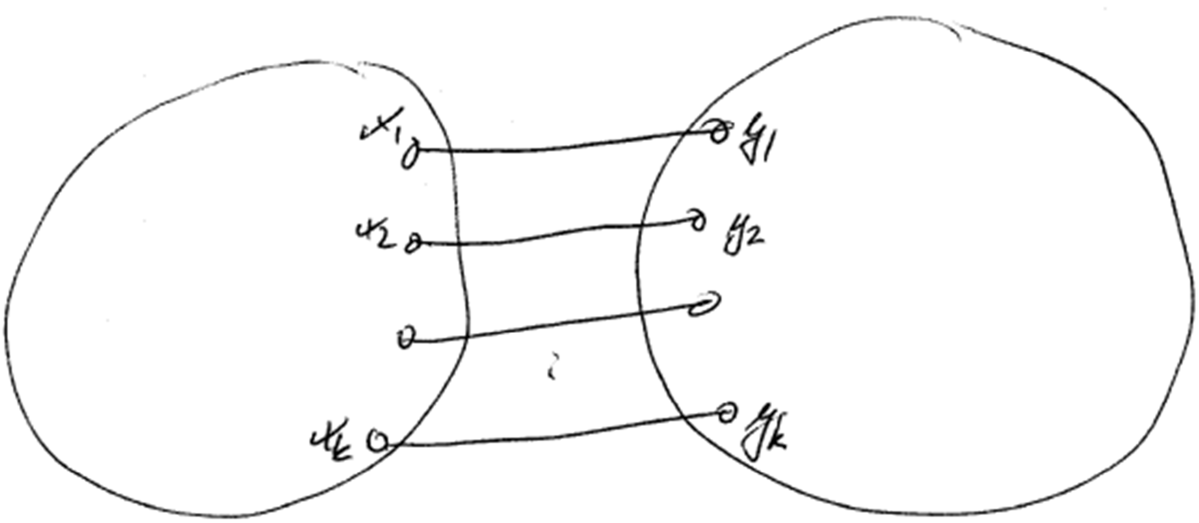


Proposition. $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ (= minimum degree)

Proof. $\kappa'(G) \leq \delta(G)$ is clear

$\kappa(G) \leq \kappa'(G)$:

Let $k := \kappa'(G)$, and let $F \subseteq E(G)$ be such that $|F| = k$ and $G \backslash F$ is disconnected.

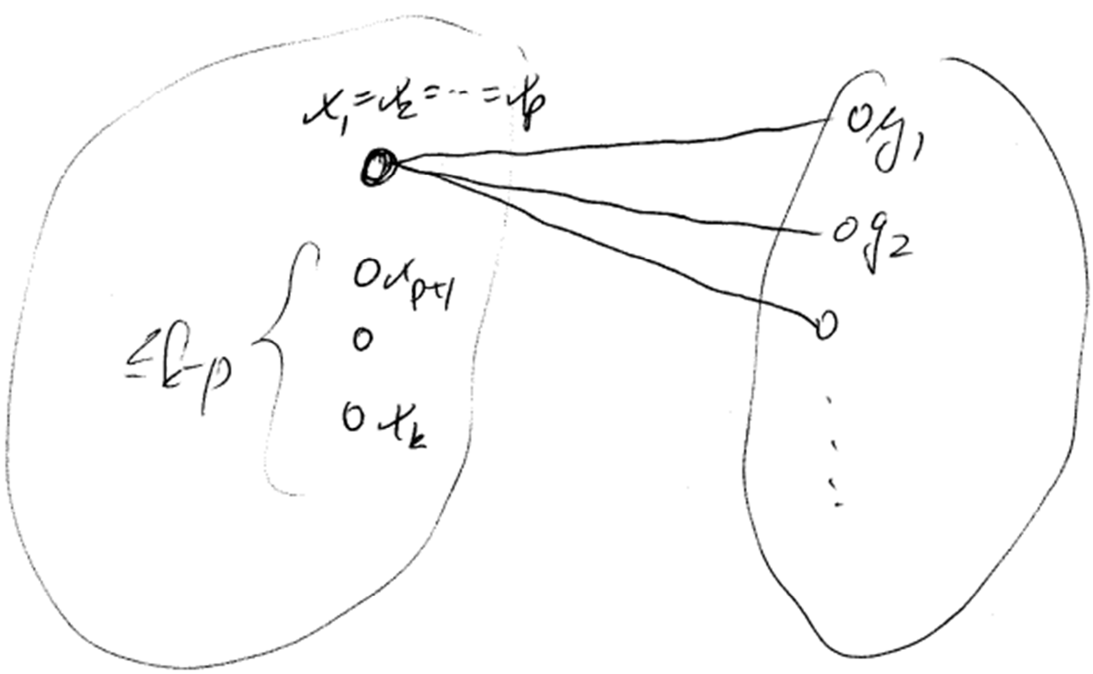


Need to find $X \subseteq V(G)$ such that $|X| \leq k$ and $G \backslash X$ is disconnected.

WMA $G \backslash \{x_1, x_2, \dots, x_k\}$ is connected. Similarly,

WMA $G \backslash \{y_1, y_2, \dots, y_k\}$ is connected. So
$V(G) = \{x_1, x_2, \dots, x_k, y_1, y_2, \dots, y_k\}$.

WMA $x_1 = x_2 = \cdots = x_p \neq x_j \quad \forall\, j = p+1, \dots, k$



Let N be the set of nbrs of x_1. Then

$|N| \leq$ # nbrs among x's+ # nbrs among y's $\leq k - p + p = k$.

If $G \backslash N$ is disconnected, then done. Otherwise

$V(G) = N \cup \{x_1\} \Rightarrow |V(G)| \leq k + 1 \Rightarrow \kappa(G) \leq k.$ $\square$

Open problem. Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\forall$ graph G $\forall\ a, b \in V(G)$ if G is $f(k)$-connected, then there exists an a- b path P such that $G \backslash V(P)$ is k-connected?

Known that $f(1) = 3$ and $f(2) = 5$.

Menger's theorem

Theorem. Let G be a multigraph, let $s, t \in V(G)$ and $S, T \subseteq V(G)$. Let $s \neq t$.

(i) The maximum number of internally disjoint s- t paths is equal to the minimum cardinality of a set $X \subseteq V(G) - \{s, t\}$ such that $G \backslash X$ has no s- t path.

(ii) The maximum number of edge-disjoint s- t paths is equal to the minimum cardinality of a set $F \subseteq E(G)$ such that $G \backslash F$ has no s- t path.

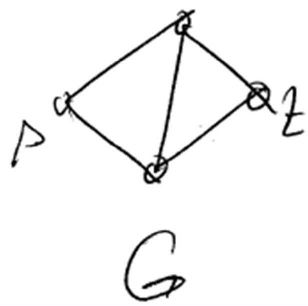
(iii) The maximum number of disjoint S- T paths is equal to the minimum cardinality of a set $W \subseteq V(G)$ such that $G \backslash W$ has no S- T path.

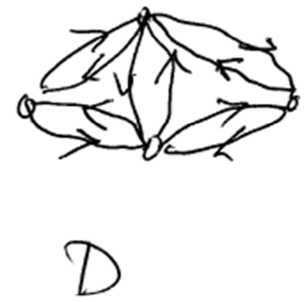
Definition. Two paths P_1, P_2 are internally disjoint if every $v \in V(P_1) \cap V(P_2)$ is an end of both.

Proof. (ii) Define a network $N = (D, s, t, c)$ as follows:

D: replace every e of G by two directed edges, one in each direction

$c(e) = 1 \ \forall \ e \in E(D)$.



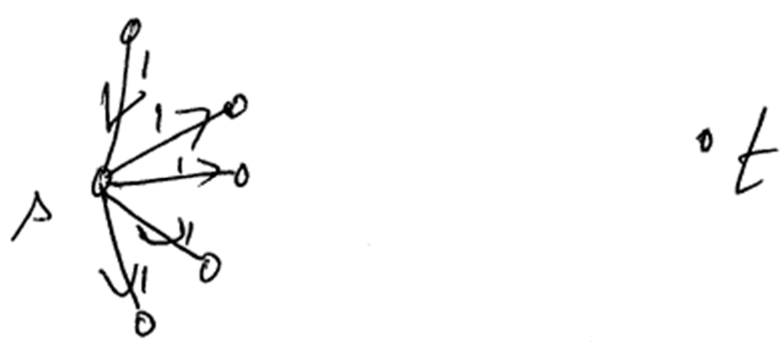


We want an integer k, a family of k edge-disjoint s- t paths and a set $F \subseteq E(G)$ of size k such that $G \backslash F$ has no s- t path.

By the MFMC theorem $\exists$ integral flow f and a cut K such that $|F| = \text{val}(f) = \text{cap}(K)$.

Let F "$:=$" K. Look at the set Z of edges e of G such that $f(e') = 1$ for one of the corresponding edges of e' of D.

Example. $\text{val}(f) = 3$.

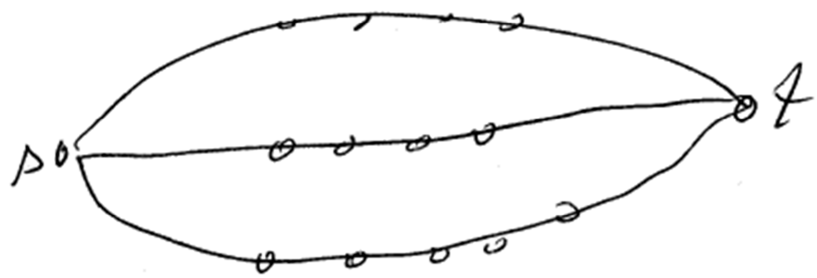


WMA D has no directed cycle consisting of edges $\{e : f(e) = 1\}$. Now Z is the set of edges of k edge-disjoint s- t paths. This proves (ii).

The rest is left as an exercise.

Food for thought.

Suppose we have 3 internally disjoint s- t paths in G.



Suppose also $\nexists$ a set $X \subseteq V(G) - \{s, t\}$ of size 3 or less such that $G \backslash X$ has no s- t path. Then, by Menger's theorem, there exist 4 internally disjoint paths.

Where are they?