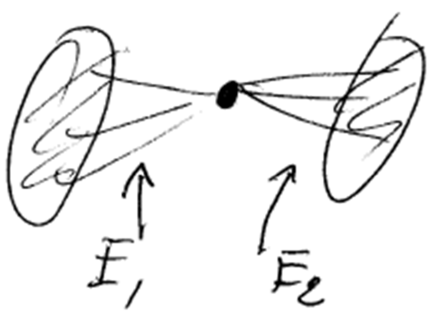
Block structure of (connected) graphs.

A ***cut-vertex*** in a multigraph G is a vertex v such that $E(G)$ can be partitioned into disjoint non-empty sets E_1, E_2 such that v is the only vertex such that both E_1 and E_2 include an edge incident with v.

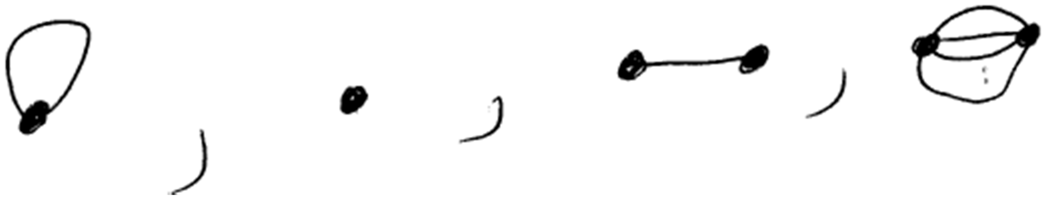
Example 1.

Example 2. $G\setminus v$ disconnected.



Definition. A ***block*** is a connected multigraph with no cut-vertex.

Examples.



Every loopless and 2-connected multigraph.

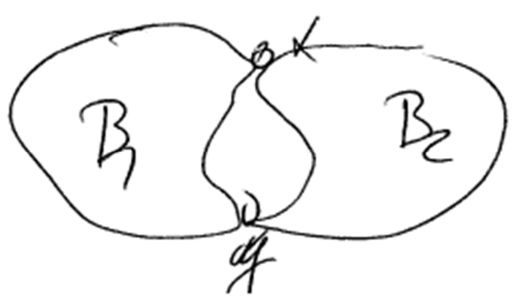
Definition. A **block of a multigraph** G is a maximal submultigraph that is a block.

Proposition. (i) Every multigraph is a union of its blocks.

(ii) If B_1, B_2 are distinct blocks of G, then $|V(B_1) \cap V(B_2)| \leq 1$, and if $v \in V(B_1) \cap V(B_2)$, then v is a cut-vertex.

Proof. (i) immediate.

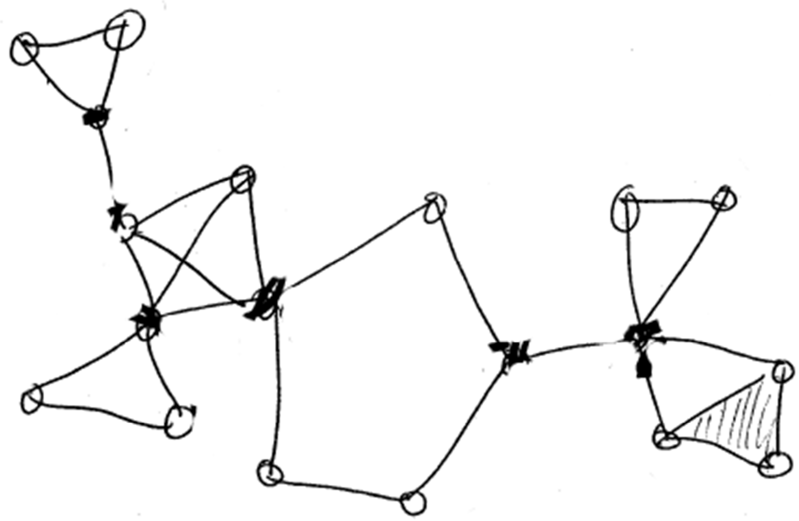
(ii) Suppose $x, y \in V(B_1) \cap V(B_2)$, $x \neq y$



$\Rightarrow B_1 \cup B_2$ is block

If B_1, B_2 are 2-connected $\Rightarrow$ $B_1 \cup B_2$ 2-connected.

Proof. Let $z \in V(B_1 \cup B_2)$. Then $B_1 \backslash z, B_2 \backslash z$ are connected, and they intersect $\Rightarrow$ $(B_1 \backslash z) \cup (B_2 \backslash z) = (B_1 \cup B_2) \backslash z$ is connected.



Given a multigraph, define a graph F as follows: $V(F) = \mathcal{B} \cup \mathcal{C}$

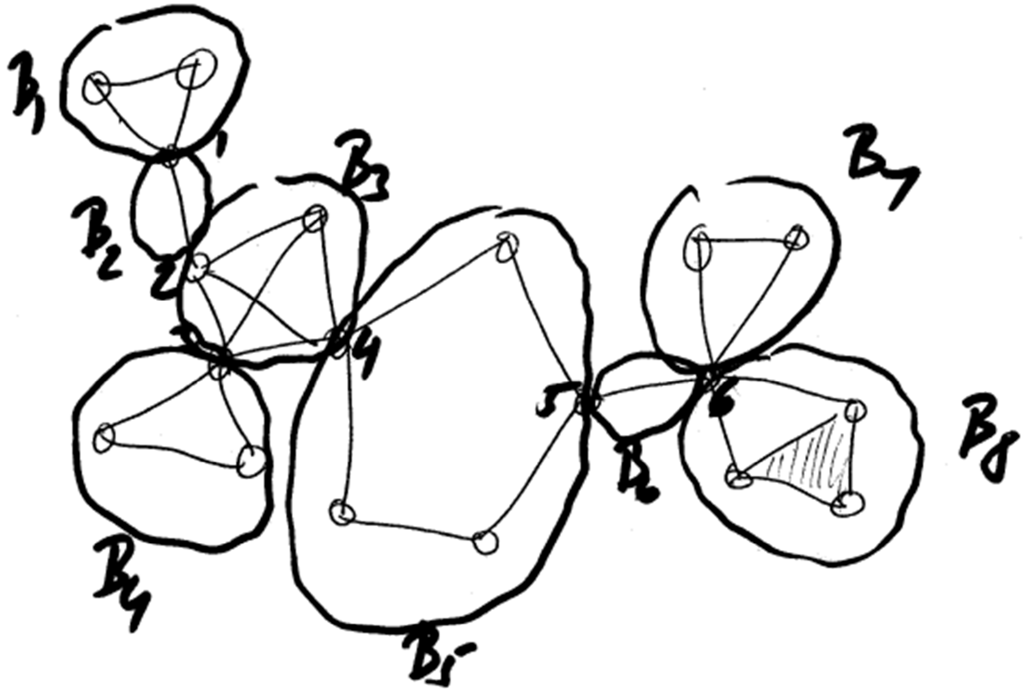
$\mathcal{B} =$ all blocks of G; $\mathcal{C} =$ all cut-vertices of G.

$B \in \mathcal{B}$ is adjacent to $c \in \mathcal{C}$ if $c \in V(B)$.

Theorem. F is a forest, and if G is connected, then it is a tree.

Definition. This is called the **block structure of a graph**.

Proof. Exercise.



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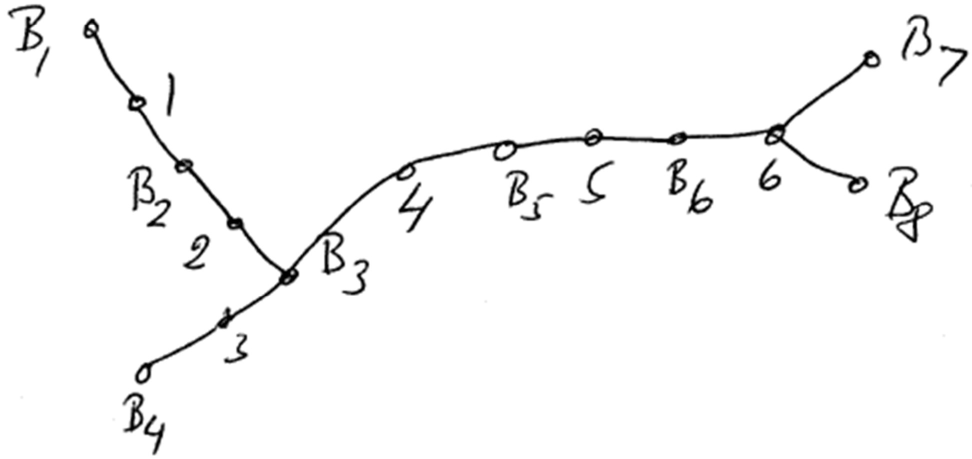
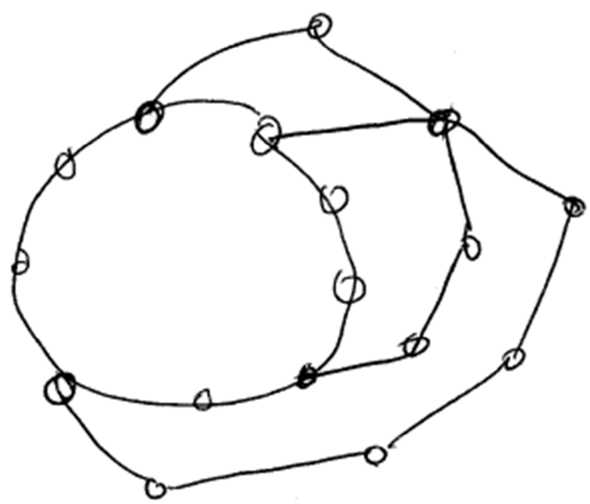


B1
1
B2
2
B3
3
B4
4
B5
5
B6
6
B7
B8

Theorem. (Ear structure of 2-connected graphs).
Let G be a 2-connected graph. Then G can be written as $G = G_0 \cup G_1 \cup \cdots \cup G_k$, where

(i) G_0 is a cycle, and
(ii) for $i = 1,2,\ldots,k$, G_i is a path with both ends in $G_0 \cup \cdots \cup G_{i-1}$, and otherwise disjoint from it.



Proof. $\exists$ cycle. We can pick $G_0, G_1, \ldots, G_k$ satisfying (i) and (ii) with k maximum.

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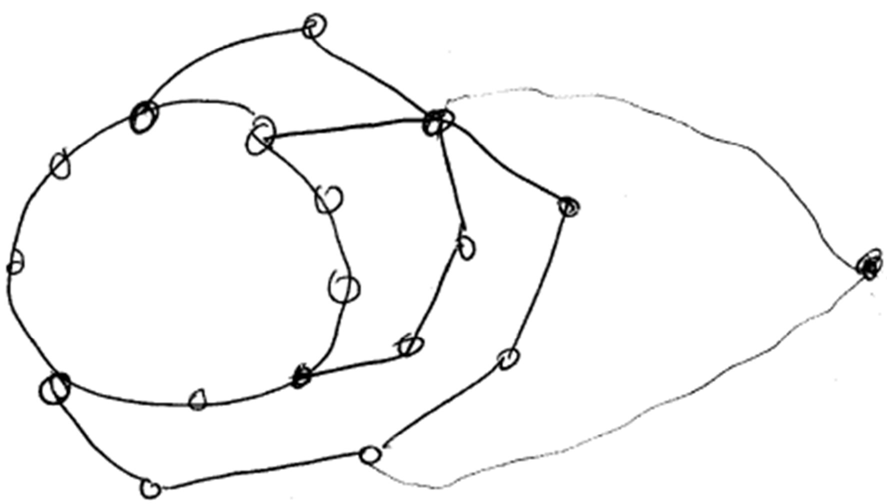


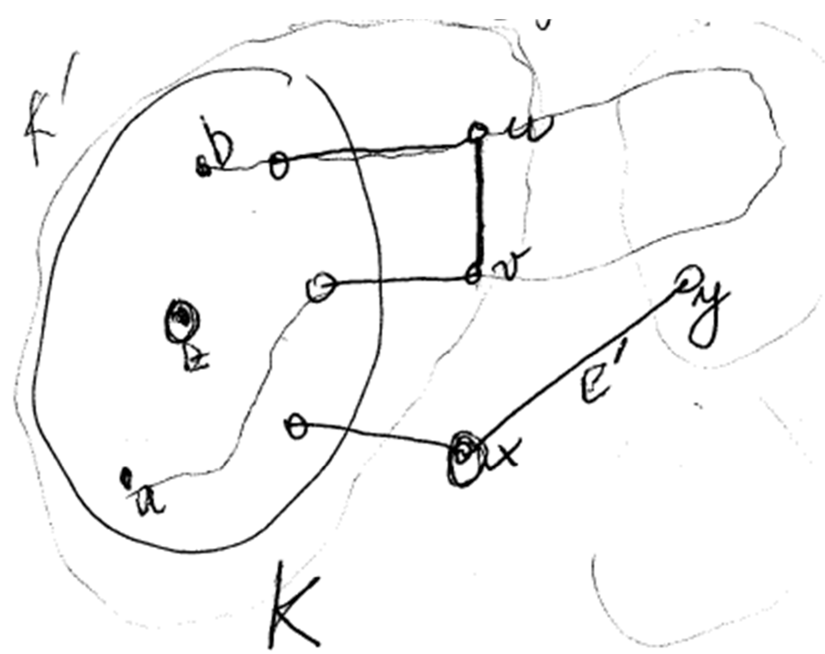
Proof. $\exists$ cycle. We can pick $G_0, G_1, \ldots, G_k$ satisfying (i) and (ii) with k maximum.

If a vertex v does not belong to $G_0 \cup \cdots \cup G_k$, then there exist paths P_1, P_2 from v to $G_0 \cup \cdots \cup G_k$, disjoint, except for v. Then $G_0, G_1, \ldots, G_k, P_1 \cup P_2$ contradicts the choice of k.

A lemma about 3-connected graphs

Theorem. Let G be a 3-connected graph on ≥ 5 vertices. Then G has an edge e such that G/e is 3-connected.

Proof. Suppose not. Thus $\forall\ e = uv\ \exists\ x \in V(G)$ such that $G \backslash \{u, v, x\}$ is disconnected. Pick $e = uv$ and x as above and a component K of $G \backslash \{u, v, x\}$ such that $|V(K)|$ is maximum. Note u has a nbr in K, for o.w. $G \backslash \{v, x\}$ is disconnected. Same for v, x. Same for other components. Let y be a nbr of x not in $V(K) \cup \{u, v\}$.



For the edge $e' = xy \ \exists \ z \in V(G)$ such that $G \backslash \{x, y, z\}$ is disconnected. Let K' be the subgraph induced by $V(K) \cup \{u, v\}$.

Claim. $K' \backslash z$ is connected.

Proof. Let $a, b \in V(K') - \{z\}$. We must show that there exists an a- b path in $K' \backslash z$. Since G is 3-connected there exists an a-b path P in $G \backslash \{z, x\}$. If P is a path in $K' \backslash z$, then we are done, and so we may assume not. Thus P leaves K' through u or v and re-enters through v or u. In either case it uses both u and v. Let P' be obtained from P by short cutting using the edge $e = uv$. Then P' is an a- b path in $K' \backslash z$, as desired. This proves the claim.

By the claim, $K' \backslash z$ is a subgraph of a component K'' of $G \backslash \{x, y, z\}$. But $|V(K'')| \geq |V(K')| - 1 \geq |V(K)| + 1$ contrary to the choice of u, v, x, K. □