G k-connected $\Rightarrow \delta(G) = \min \text{degree} \geq k$

Converse?

Theorem (Mader) Every graph of minimum degree at least $4k$ has a k-connected subgraph.

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Thoughts about a possible proof. Let $n := |V(G)|$

1. Replace minimum degree by average degree
   So $4k \leq$ average deg $= \sum_v \deg(v)/n = 2|E(G)|/n$,
   which is the same as $|E(G)| \geq 2kn$

2. Let's try to prove that for some suitable c, β
   $(*)$ $|E(G)| \geq ckn + \beta + 1 \Rightarrow G$ has k-connected subgraph.

Is it possible that for some values of c, β:

(#) If G is a counterexample, then so is G_1or G_2?

If not, then $|E(G)| \geq ckn + \beta + 1$ and $|E(G_1)| \leq ckn_1 + \beta$ and $|E(G_2)| \leq ckn_2 + \beta$.

But these conditions could be contradictory

$$|E(G)| \le |E(G_1)| + |E(G_2)| \le ckn_1 + \beta + ckn_2 + \beta \le$$

$$\le ck(n+k) + 2\beta = ckn + \beta + ck^2 + \beta \le ckn + \beta$$

if $ck^2 + \beta \le 0$.

So if $ck^2 + \beta \le 0$, then above computation gives a contradiction, and hence proves (#).
If we pick $\beta := -ck^2$, then we have an inductive proof of (*):

$$|E(G)| \ge ckn - ck^2 + 1 \Rightarrow G \text{ has } k\text{-connected subgraph.}$$

But how about the base of the induction?
$n = k + 1$? No way.

We should choose the base so that the edge bound will force the graph to be complete: $ckn - ck^2 + 1 \ge n^2/2$. Best chance for $n = ck$.

Thus we are trying to prove

(**) If $|V(G)| \ge ck$ and $|E(G)| \ge ckn - ck^2 + 1$, then G has a k-connected subgraph

Base case: If we choose suitable c, then there is no graph G with $|V(G)| = ck$ and $|E(G)| \geq ckn - ck^2 + 1$: If $|V(G)| = ck$, then

$$(ck)^2 - ck^2 + 1 \leq |E(G)| \leq \binom{ck}{2} < \frac{1}{2}(ck)^2$$

$$\frac{1}{2}(ck)^2 < ck^2$$

$$c < 2$$

If $c \geq 2$, then the graph does not exist.

Induction step: Same as before, except we need to show that $|V(G_i)| \geq ck$. That follows from

Lemma. If G is a minimum counterexample to $(**)$, then $\delta(G) \geq ck$.

Proof. We have already shown that $|V(G)| > ck$.

If v has degree $\leq ck$, then

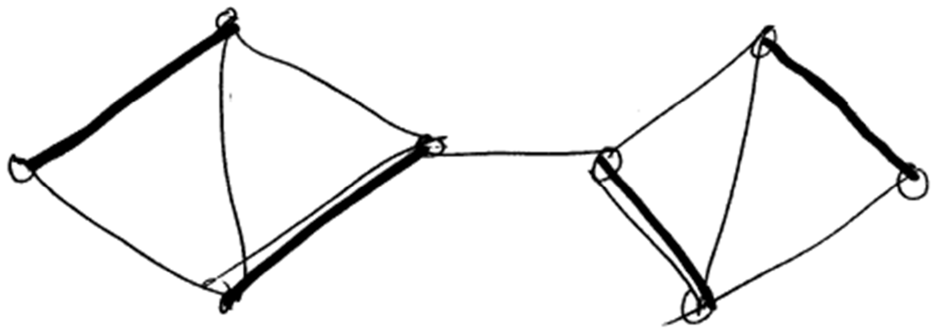
$$|E(G \backslash v)| \geq ckn - ck^2 + 1 - ck = ck(n-1) - ck^2 + 1 =$$

$$= ck(|V(G \backslash v)| - 1) - ck^2 + 1$$

$\Rightarrow G \backslash v$ is a smaller counterexample. $\square$

Matching

A **matching** in G is a set $M \subseteq E(G)$ such that every vertex of G is incident with at most one edge of M.



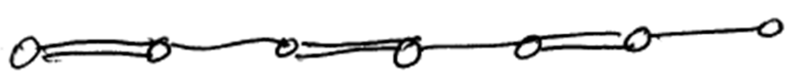
M **saturates** $v \in V(G)$ or v is **saturated** by M if v is incident with an edge of M.

A matching is **perfect** if it saturates every vertex.

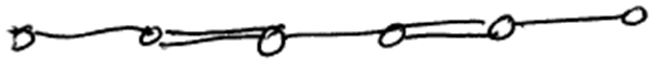
A **maximum matching** is a matching M in G such that there is no matching M' with $|M'| > |M|$.

A **maximal matching** is a matching M such that there is no matching M' with $M \subsetneq M'$.

An M-**alternating path** is a path P such that edges in M and edges not in M alternate along P.



An M-**augmenting path** is an M-alternating path P that starts and ends in an M-unsaturated vertex.



Let $M' := M \Delta E(P)$. Then M' is a matching with $|M'| > |M|$.

Theorem. (Berge) A matching M in G is maximum if and only if there is no M-augmenting path.

Proof. $\Rightarrow$ done

$\Leftarrow$ Assume M is not maximum. Let M' be a matching with $|M'| > |M|$. Look at the graph H with $V(H) = V(G)$, $E(H) = M \triangle M'$.

Then $\triangle(H) \leq 2$. The components of H are:

- even cycles (same number of edges of M and M')
- paths

$\Rightarrow \exists$ component P of H that has more edges in M' than in M. That's an M-augmenting path. $\square$