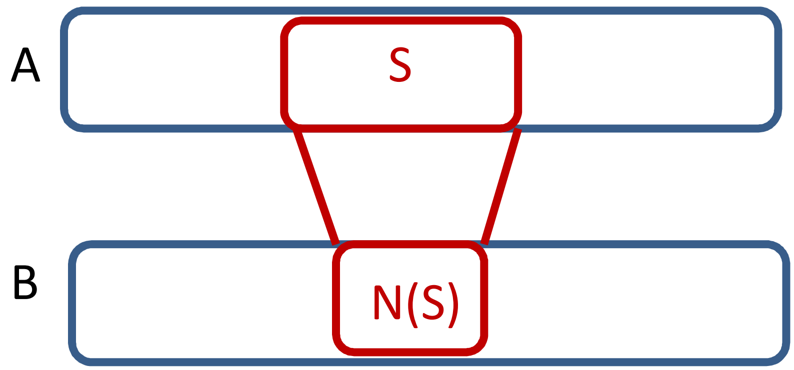
Matchings in bipartite graphs

Let G be a bipartite graph with bipartition (A, B). A matching M is a **complete matching from** A **to** B if it saturates every vertex of A.

If $|A| = |B|$, then a complete matching from A to B is the same as a perfect matching.

Obstruction:



$N(S) := \{v \notin S: v \text{ is adjacent to a vertex in } S\}$

If $|N(S)| < |S|$ for some $S \subseteq A$, then $\nexists$ complete matching A to B

Theorem. (Hall) A bipartite graph with bipartition (A, B) has a complete matching from A to B if and only if $|N(S)| \geq |S|$ for every $S \subseteq A$.

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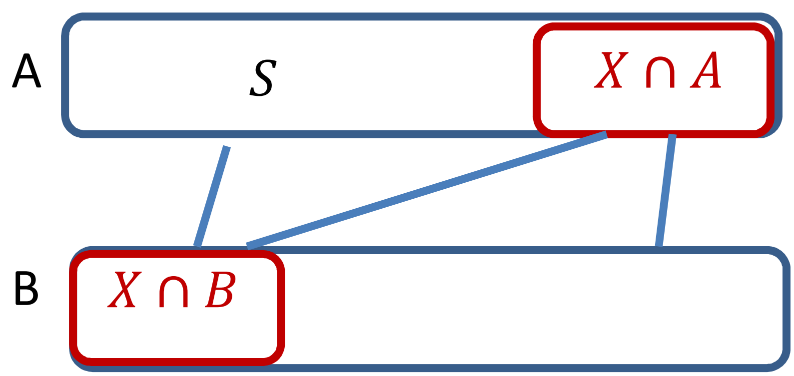
Proof #1. Using Menger's theorem

$\Rightarrow$ already done

$\Leftarrow$ If there exist $|A|$ disjoint paths from A to B, then their edge-sets form a complete matching from A to B. Thus WMA $\nexists |A|$ disjoint A-B paths. By Menger's theorem $\exists X \subseteq V(G)$ such that $G \backslash X$ has no A-B path and $|X| < |A|$.



Let $S := A - X$. Then $N(S) \subseteq X \cap B$ and hence

$$|N(S)| \leq |X \cap B| = |X| - |X \cap A| < |A| - |X \cap A| = |S|,$$

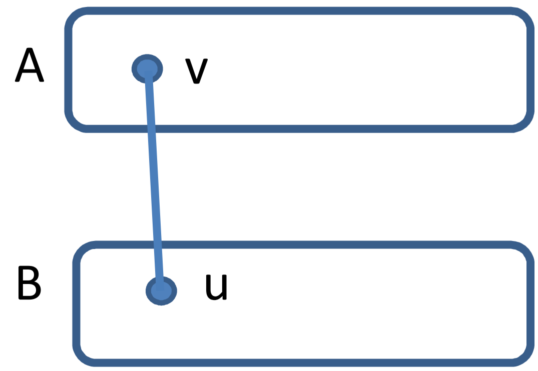
a contradiction. $\square$

Proof #2. From first principles ($\Leftarrow$ only)

Case 1. $|N(S)| > |S|$ for every $\emptyset \neq S \subsetneq A$.

Pick $v \in A$ and a neighbor u of v.

Apply induction to $G \backslash \{u, v\}$.

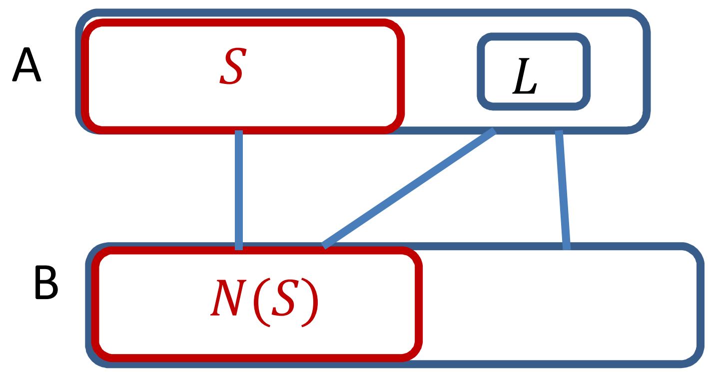


Case 2. $|N(S)| = |S|$ for some $\emptyset \neq S \subsetneq A$.

Let $G_1 := G[S \cup N(S)]$

$G_2 := G \backslash (S \cup N(S))$

Apply induction to G_1 and G_2.



G_1 clearly satisfies the induction hypothesis.

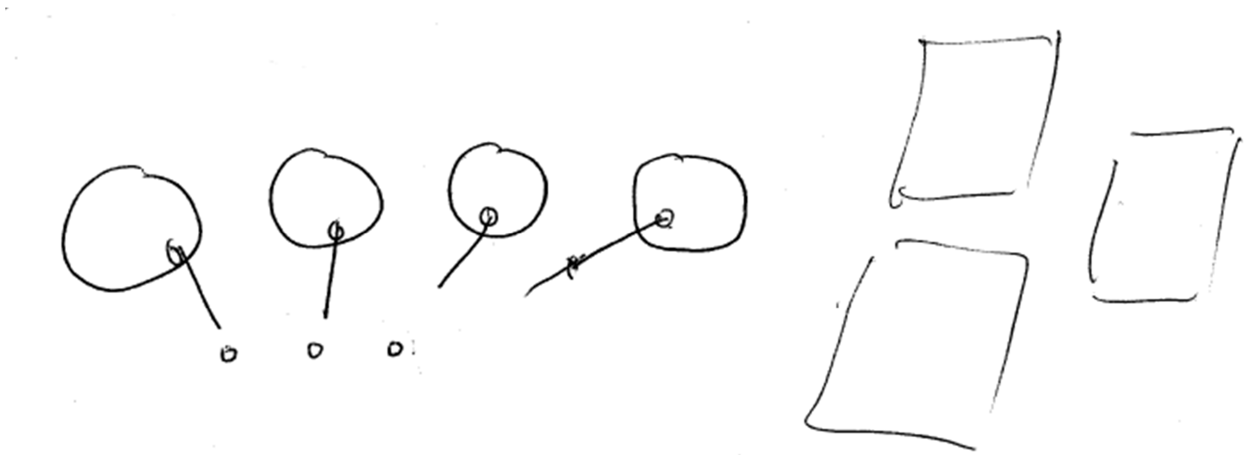
To see that G_2 satisfies the induction hypothesis for $L \subseteq A - S$, look at $N_G(L \cup S)$.

$$|N_G(S)| + |N_{G_2}(L)| = |N_G(S \cup L)| \geq |S \cup L| = |S| + |L|$$

and so $\left|N_{G_2}(L)\right| \geq |L|$, as desired. $\square$

Perfect matchings in (not necessarily bipartite) graphs

An obstruction:



Let $o(H) :=$ # of odd components of H.

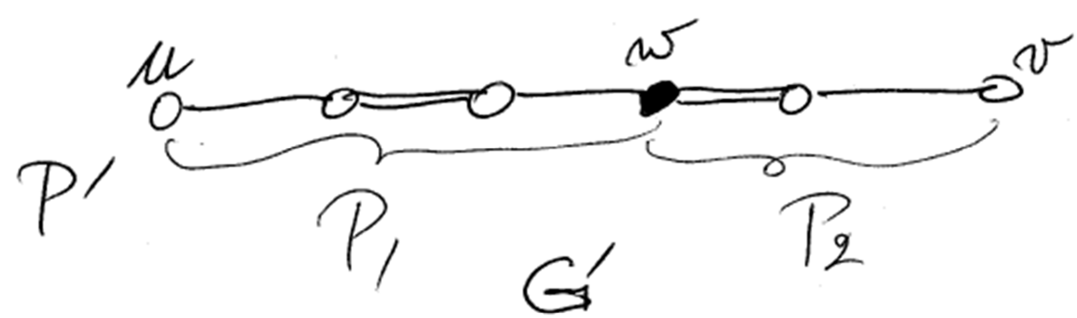
If $o(G \backslash X) > |X|$ for some X, then G has no perfect matching.

Tutte's 1-factor theorem (1947). A graph G has a perfect matching if and only if $o(G \backslash X) \leq |X|$ for every $X \subseteq V(G)$.

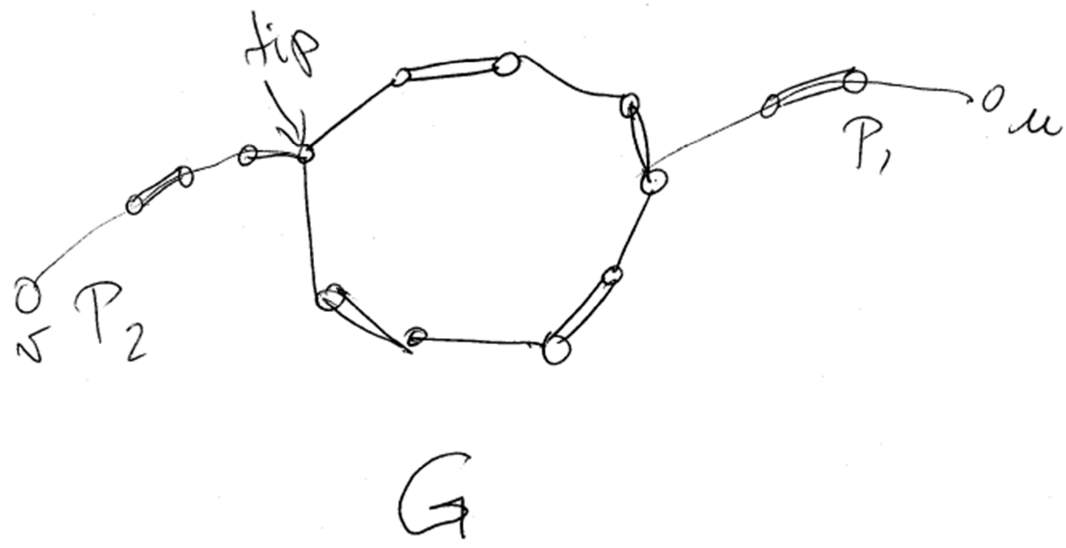
Definition. Let M be a matching in G. A cycle C in G of length $2k+1$ containing k edges of M is called an M-**blossom**. Let G/C denote the graph obtained from G by contracting all edges of C and deleting all loops and parallel edges.

Lemma. Let M be a matching in G, and let C be an M-blossom in G. Let $G' := G/C$ and $M' := M - E(C)$. If M is a maximum matching in G, then M' is a maximum matching in G'.

Proof. Suppose not. Then $\exists M'$-augmenting path P' in G'. We will exhibit an M-augmenting path in G. Let w be the new vertex of G'. WMA $w \in V(P')$, for otherwise P' is as desired.



The vertex w divides P' into P_1 and P_2. Let u, v be the ends of P'. WMA by symmetry that the edge of P_2 incident with w is in M.



Then in G the path P_2 becomes a path from v to the tip of the blossom, and P_1 becomes a path from u to the blossom. Follow P_1 from u to $u' \in V(C)$, then follow C along the even path from u' to the tip, and then follow P_2. That gives an M-augmenting path in G. $\square$

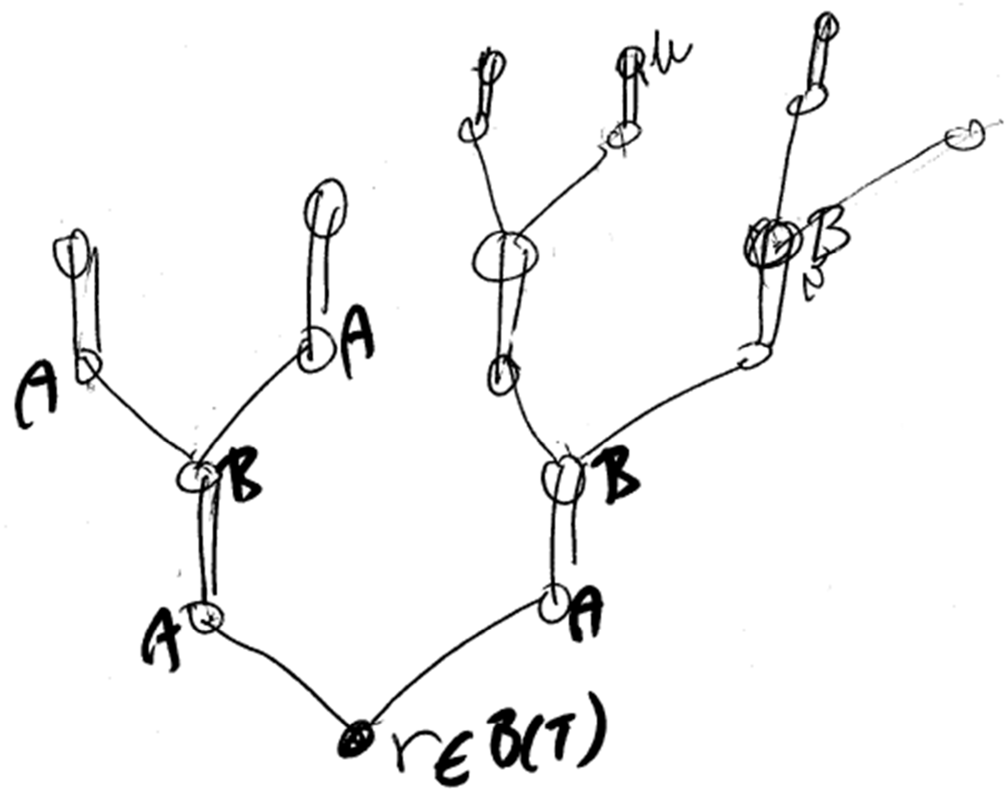
Definition. Let M be a matching in G, and let $r \in V(G)$ be M-unsaturated. An M-**alternating tree rooted at** r is a tree T such that

(i) T is a subgraph of G

(ii) $r \in V(T)$

(iii) every path in T with end r is M-alternating

(iv) if $e \in M$ is incident with a vertex of T, then $e \in E(T)$



Given an M-alternating tree T let

$$A(T) := \{v \in V(T) : v \text{ is at odd distance from } r \text{ in } T\}$$

$$B(T) := \{v \in V(T) : v \text{ is at even distance from } r \text{ in } T\}$$

Theorem. Let G be a graph, let M be a maximum matching in G, let $r \in V(G)$ be M-unsaturated, and let T be an M-alternating tree rooted at r. Then there exists a set $X \subseteq V(G)$ such that $X \cap V(T) \subseteq A(T)$ and $o(G \backslash X) > |X|$.

Note that this implies Tutte's theorem.