Nowhere-zero flows

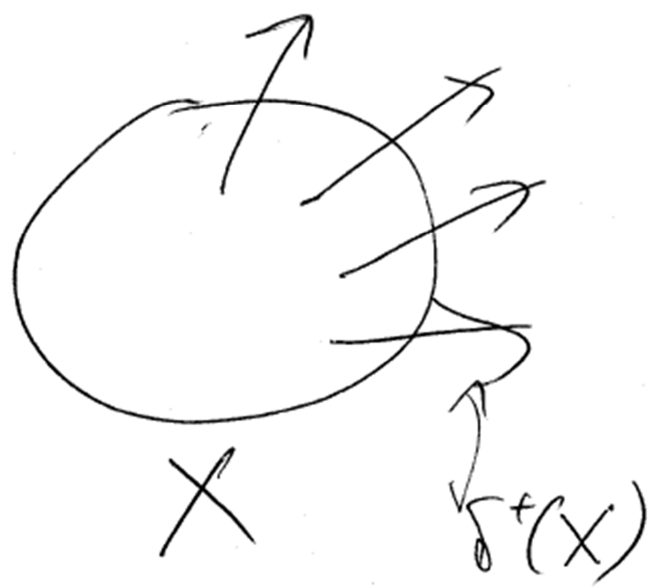
Let D be a digraph, Γ Abelian group. A Γ-circulation in D is a mapping $f: E(D) \to \Gamma$ such that

$$f^+(v) = f^-(v),$$

where $f^+(v) = \sum_{e \in \delta^+} f(e)$, $f^-(v) = \sum_{e \in \delta^-} f(e)$ and

$$\delta^+(X) = \{e \in E(D)\text{: tail in } X\text{, head in } V(D) - X\}$$

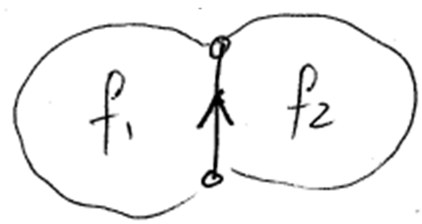
$$\delta^-(X) = \{e \in E(D)\text{: tail in } X\text{, head in } V(D) - X\}$$



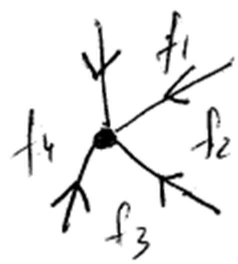
A **nowhere-zero** Γ-flow is a Γ-circulation such that $f(e) \neq 0$ for every $e \in E(D)$. A **nowhere-zero** k-flow is a Z-circulation f such that $0 < |f(e)| < k$ for every edge e. Compare to nowhere-zero (NZ) Z_k-flow. These are properties of the underlying undirected graph of D.

Thm. Let G be a plane graph. Then G has a NZ k-flow if and only if G is face k-colorable.

Pf. $\Leftarrow$ Color the faces using $1, \ldots, k$. Let D be an orientation of G. Define $\phi(e) = c(f_1) - c(f_2)$.



Then $\phi(e) \neq 0 \ \forall\ e \in E(D)$.



$\Rightarrow$ Any integer-valued circulation in a plane graph is an integer linear combination of "facial circulations"

Now let ϕ be a NZ k-flow. Then there exists a function $\beta: F(G) \to \mathbb{Z}$ such that

$$\phi(e) = \beta(f_1) - \beta(f_2),$$

where f_1 is the face to the left of e and f_2 is the face to the right. Define $\alpha(f)$ to be the residue class of $\beta(f)$ (mod k). Then α is a k-coloring of the faces.

Thm. Let D be a digraph. There is a polynomial P such that for every Abelian group Γ the number of NZ Γ-flows in D is $P(|\Gamma|)$.

Proof. If D has no non-loop edge, then $P(x) = (x-1)^{|E(D)|}$.

Otherwise pick a non-loop edge e, and let $\phi(D)$ be the # of NZ Γ-flows in D. Then

$$\phi(D) = \phi(D/e) - \phi(D \backslash e)$$

Theorem follows by induction. □

Thm. A graph has $\mathbb{Z}_k$-flow if and only if it has a k-flow.

Pf. $\Leftarrow$ easy.

$\Rightarrow$ Let $f: E(D) \to \mathbb{Z}$ be such that

(1) $0 < |f(e)| < k \ \forall\, e \in E(D)$

(2) $f^+(v) \equiv f^-(v) (\text{mod}\, k)$

Let $D(f) \coloneqq \sum_{v \in V(D)} |f^+(v) - f^-(v)|$, and choose f satisfying (1) and (2) with $D(f)$ minimum. WMA $f(e) > 0\ \forall\ e \in E(D)$. Let

$$A = \{v \in V(D) \colon f^+(v) > f^-(v)\}$$

and

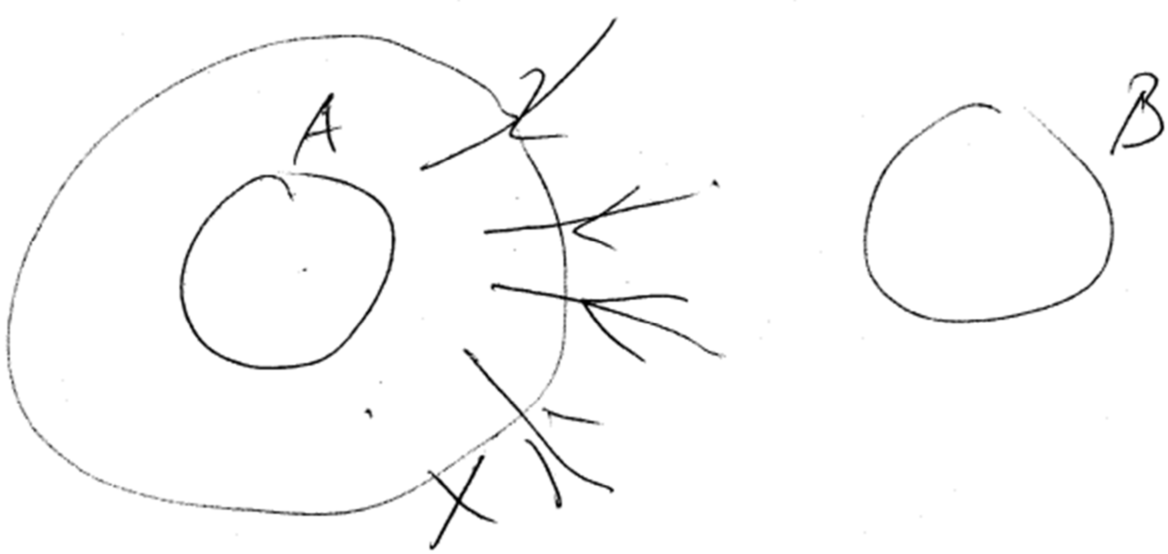
$$B = \{v \in V(D) \colon f^+(v) < f^-(v)\}$$

Claim. $\nexists$ directed $A \to B$ path.

Pf. O.w. decrease the flow by k along such path. If $A = B = \emptyset$, then done, so WMA one is empty, and hence both are, because

$$\sum_{v \in V(D)} f^+(v) = \sum_{e \in E(D)} f(e) = \sum_{v \in V(D)} f^-(v)$$

By the claim $\exists\ X$ with $A \subseteq X$, $B \cap X = \emptyset$ and $\delta^+(X) = \emptyset$.



$$\sum_{v\in X} f^+(v) > \sum_{v\in X} f^-(v)$$

$$-\sum_{e\in\langle X\rangle} f(e) + \sum_{v\in X} f^+(v) > -\sum_{e\in\langle X\rangle} f(e) + \sum_{v\in X} f^-(v)$$

$0 = f^+(X) > f^-(X)$, a contradiction. $\square$

Corollary. For a graph G and a group Γ, the following are equivalent:

(1) G is NZ Γ-flow

(2) G has a NZ k-flow, where $k = |\Gamma|$.

Corollary. A cubic graph has a NZ 4-flow if and only if it is 3-edge-colorable.

Proof. NZ 4-flow ⇔ NZ $Z_2 \times Z_2$-flow ⇔ 3-edge-coloring using

the colors $(0,1), (1,0), (1,1)$

Thm. If G is plane, then G has a NZ k-flow if and only if G^* is k-colorable

Corollary. The 4CT is equivalent to: Every 2-edge-connected cubic planar graph is 3-edge-colorable.

Thm. A cubic graph has a NZ 3-flow $\Leftrightarrow$ it is bipartite.

Pf. NZ 3-flow $\Leftrightarrow \mathbb{Z}_3$-flow $\Leftrightarrow \exists$ orientation s.t. $f = 1$ is a $\mathbb{Z}_3$-flow. Since G cubic $\Rightarrow$ sources vs. sinks is a bipartition. That proves $\Rightarrow$.

$\Leftarrow$ Direct one way $\Rightarrow \mathbb{Z}_3$-flow $\square$

3-flow conjecture. Every 4-edge-connected graph has a NZ 3-flow.

3-edge-coloring conjecture. Every 2-edge-connected cubic graph with no Petersen minor is 3-edge-colorable ($\Leftrightarrow$ NZ 4-flow).

4-flow conjecture. Every 2-edge-connected graph with no Petersen minor has a NZ 4-flow.

This implies

Grőtzsch conjecture. Let G be a planar graph of max degree 3 with no subgraph H s.t. H has all vertices of degree 3, except for exactly one of degree 2. Then G is 3-edge-colorable.

Implies the 4CT.

5-flow conjecture. Every 2-edge-connected graph has a NZ 5-flow.