INTRODUCTION TO PLANAR GRAPHS

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ABSTRACT

The topic of planar graphs is covered in many books and articles, but the treatment usually relies on intuition or on deep topological theorems that are quoted without proof. I give a self-contained rigorous introduction to planar graphs.

1 The Jordan curve theorem and Euler's formula

Throughout these notes *graphs* are allowed to have loops and multiple edges.

Definition 1.1. A *polygonal arc* is a set $A \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the interval $[0, 1]$. The images of 0 and 1 under the homeomorphism are called the *ends* of A. A *polygon* is a set $B \subseteq \mathbb{R}^2$ which is the union of finitely many straight line segments and is homeomorphic to the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. By a *bend* of a polygonal arc or a polygon P we mean a point of P where two different straight line segments meet. Thus P has finitely many bends.

Definition 1.2. Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and for $x, y \in \Omega$ let us define $x \sim y$ if there exists a polygonal arc $A \subseteq \Omega$ with ends x and y. Then $\sim$ is an equivalence relation, and the equivalence classes are called the *arcwise connected components of* Ω. If $x \sim y$ for any two $x, y \in \Omega$, then we say that Ω is *arcwise connected*. Now if $F \subseteq \mathbb{R}^2$ is closed, we say that an arcwise connected component of $\mathbb{R}^2 - F$ is a *face* of F.

Theorem 1.3 (Jordan curve theorem for polygons)**.** *Every polygon P has exactly two faces, of which exactly one is bounded. The boundary of each of the two faces is P.*

Proof. For $x \in \mathbb{R}^2 - P$ and a half-line L originating in x and containing no bends of P let $\pi(x, L)$ denote the number of intersections of L with P modulo 2. It is easy to check that this can be extended to all half-lines originating in x in such a way that $\pi(x, L)$ does not depend on L. Let us call that value $\pi(x)$. Furthermore, it follows that the function π is continuous, and hence is constant on each arcwise connected component of $\mathbb{R}^2 - P$. By choosing two points x_1 and x_2 close

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to each other, but on opposite sides of a straight-line segment of P we can show that $\pi(x_1) \neq \pi(x_2)$. Thus P has at least two faces.

Suppose now that P has at least three faces, and choose a point in each, say x_1, x_2, x_3. Pick a point x on the boundary of P inside a straight-line segment S of P. Pick a small open neighborhood O of x such that $O \cap P = O \cap S$. By shooting a half-line from each of x_1, x_2, x_3 toward P and then following the boundary of P until we hit O we see that each of x_1, x_2, x_3 can be reached from a point in O by a polygonal arc not intersecting P. But $O - P$ has at most two arcwise connected components, and hence P has at most two faces. It follows that every point of $O \cap S$ belongs to the boundary of both faces of P, and since x was arbitrary, we deduce that the boundary of both faces of P is P. We leave it as an exercise to show that exactly one of the faces is bounded. □

Definition 1.4. A *plane graph* is graph G such that

(i) $V(G) \subseteq \mathbb{R}^2$,

(ii) for every non-loop edge $e \in E(G)$ with ends u and v there exists a polygonal arc A with ends u and v such that $e = A - \{u, v\} \subseteq \mathbb{R}^2 - V(G)$,

(iii) for every loop $e \in E(G)$ incident with $u \in V(G)$ there exists a polygon P containing u such that $e = P - \{u\} \subseteq \mathbb{R}^2 - V(G)$, and

(iv) if $e, e' \in E(G)$ are distinct, then $e \cap e' = \emptyset$.

Thus an edge of a plane graph is a subset of the plane that includes no vertices, not even its ends. We define the *point set of* G to be the set $\bigcup_{e\in E(G)} e \cup V(G)$, and by abusing notation we shall denote this set also by G. Thus the faces of G are the arcwise connected components of $\mathbb{R}^2 - G$. The set of faces of a plane graph G will be denoted by $F(G)$.

Definition 1.5. A graph G is *planar* if it is isomorphic to a plane graph Γ. We say that Γ is a (planar) drawing of G.

Lemma 1.6. *Let G be a plane graph, let $e \in E(G)$, and let $x_1, x_2 \in \mathbb{R}^2 - G$ be two points such that the straight line segment connecting them intersects e exactly once, and is otherwise disjoint from G. Let f_i be the face of G that includes x_i. Then e is a subset of the boundary of both f_1 and f_2 and is disjoint from the boundary of every other face of G.*

Proof. This follows by a similar argument as the second half of the proof of Theorem 1.3. □

Corollary 1.7. *If G is a plane graph and $f \in F(G)$, then the boundary of f is the point set of a subgraph of G. In particular, if the boundary of f includes a point belonging to an edge $e \in E(G)$, then it includes the entire edge e.*

Definition 1.8. Let G, e, f_1, f_2 be as in Lemma 1.6. We will refer to f_1 and f_2 as the two faces *incident* with e. Please note that f_1 and f_2 need not be distinct.

Lemma 1.9. *Let G be a plane graph that is a forest. Then G has exactly one face.*

Proof. Exercise. Use induction on the number of vertices plus the number of bends. □

Lemma 1.10. *Let G' be a subgraph of a plane graph G. Then*

(i) *every face of G is a subset of a face of G',*

(ii) *if f is a face of G and $bd(f) \subseteq G'$, then f is a face of G', and*

(iii) *if $f' \in F(G')$ is disjoint from G, then $f' \in F(G)$.*

Proof. Statement (i) follows immediately: a face f of G is an arcwise-connected subset of $\mathbb{R}^2 - G$, and hence is a subset of an arcwise-connected component of $\mathbb{R}^2 - G'$, that is, a face of G'.

To prove (ii) let f be a face of G. By (i) there exists a face f' of G' such that $f \subseteq f'$. If $f = f'$, then (ii) holds, and so we may assume that the inclusion is proper. Hence there exists a point $x' \in f' - f$. Let $x \in f$, and let $A \subseteq f'$ be a polygonal arc with ends x, x'. Since A is not a subset of f, it intersects the boundary of f, say in a point z. But $z \in f'$, and hence $z \notin G'$, contrary to hypothesis. This proves (ii).

To prove (iii) let $f' \in F(G')$ be disjoint from G. Then f' is an arcwise connected subset of $\mathbb{R}^2 - G$, and therefore is a subset of a face f of G. By (i) we have $f \subseteq f'$, and hence $f' = f \in F(G)$, as desired. □

Lemma 1.11. *Let e be an edge of a plane graph G that belongs to a cycle of G, and let $f_1, f_2 \in F(G)$ be the two faces incident with e. Then $f_1 \neq f_2$.*

Proof. For $i = 1, 2$ the face f_i is a subset of a face f_i' of C, by Lemma 1.10(i). By Lemma 1.6 the edge e is disjoint from the boundary of all faces of $F(G) - \{f_1, f_2\}$, and hence it is disjoint from the boundary of all faces of $F(C) - \{f_1', f_2'\}$. But the edge e belongs to the boundary of both faces of C by Theorem 1.3, and hence $f_1' \neq f_2'$, implying that $f_1 \neq f_2$, as desired. □

Lemma 1.12. *Let G be a plane graph, let $e \in E(G)$, and let f_1 and f_2 be the two faces incident with e. Let f_{12} denote the point set $f_1 \cup e \cup f_2$. Then f_{12} is a face of $G\backslash e$ and $F(G) - \{f_1, f_2\} = F(G\backslash e) - \{f_{12}\}$.*

Proof. To prove the first assertion we notice that f_{12} is arcwise connected, and hence is a subset of a face f' of $G\backslash e$. We may assume that f_{12} is a proper subset of f', for otherwise $f_{12} \in F(G\backslash e)$, as desired. Thus there exists a polygonal arc A with ends $x \in f_{12}$ and $y \in f' - f_{12}$ such that $A \subseteq f'$. By considering a proper subset of A we may assume that $A \cap e = \emptyset$. Since $y \in f' - f_{12} \subseteq \mathbb{R}^2 - G - f_1 - f_2$ it follows that y belongs to a face of G other than f_1 or f_2, but then A connects points belonging to different faces of G, and yet $A \cap G = \emptyset$, a contradiction. Thus $f_{12} \in F(G\backslash e)$.

For the second assertion let $f \in F(G) - \{f_1, f_2\}$. Since f_1, f_2 are the only two faces of G incident with e by Lemma 1.6, we deduce that $\mathrm{bd}(f) \subseteq G\backslash e$, and hence $f \in F(G\backslash e)$ by Lemma 1.10(ii). Clearly $f \neq f_{12}$. Conversely, let $f' \in F(G\backslash e) - \{f_{12}\}$. Then $f' \cap G = \emptyset$, and hence $f' \in F(G)$ by Lemma 1.10(iii). Further, $f' \notin \{f_1, f_2\}$, because $f_1, f_2 \notin F(G\backslash e)$. □

Theorem 1.13 (Euler's formula)**.** *Every connected plane simple graph G satisfies $|V(G)| + |F(G)| = |E(G)| + 2$.*

Proof. We proceed by induction on $|E(G)|$. If G has no cycles, then it is a tree, and the formula holds by Lemma 1.9. Thus we may assume that G has a cycle C. Let $e \in E(C)$, and let $f_1, f_2 \in F(G)$ be the two faces incident with e. Then $f_1 \neq f_2$ by Lemma 1.11. The formula follows from Lemma 1.12 by induction applied to the graph $G\backslash e$. □

Corollary 1.14. *Every simple planar graph G on $n \geq 3$ vertices has at most $3n-6$ edges. Moreover, if G has no triangles, then it has at most $2n - 4$ edges.*

Proof. We may assume that G is connected, for an edge joining vertices in different components of G may be added without violating planarity. Let q be the number edge-face incidences (e, f), with the proviso that if f_1 and f_2 are the two faces incident with e and $f_1 = f_2$, then the incidence $(e, f_1) = (e, f_2)$ is counted twice. Then $q = 2|E(G)|$. On the other hand, since G has no loops or parallel edges, each face contributes at least three toward q, and hence $q \geq 3|F(G)|$. Thus $|F(G)| \leq 2|E(G)|/3$, and substituting this into Euler's formula gives the the first inequality. The second inequality follows similarly. □

Corollary 1.15. *The graphs K_5 and $K_{3,3}$ are not planar.*

Proof. This follows immediately from Corollary 1.14. □

2 Kuratowski's theorem

A *subdivision* of a graph H is a graph obtained from H replacing the edges of H by internally disjoint paths of nonzero length with the same ends. An H subdivision in a graph G is a subgraph of G isomorphic to a subdivision of H. From Corollary 1.15 we deduce

Corollary 2.1. *No planar graph has a* K_5 *or* $K_{3,3}$ *subdivision.*

The objective of this section is to prove Kuratowski's theorem, the converse of Corollary 2.1.

Lemma 2.2. *Let* G *be a plane graph consisting of two vertices and three internally disjoint paths* P_1, P_2, P_3 *joining them. Then* G *has precisely three faces with boundaries* $P_1 \cup P_2$*,* $P_2 \cup P_3$ *and* $P_1 \cup P_3$*, respectively.*

Proof. The graph $P_1 \cup P_2$ has exactly two faces by Theorem 1.3; let f_3 be the one disjoint from P_3. Let f_1, f_2 be defined similarly. Then $f_1, f_2, f_3 \in F(G)$ by Lemma 1.10(iii), and they are distinct, because their boundaries are distinct. Conversely, let f be a face of G, and let $x \in \mathrm{bd}(f) - V(G)$. From the symmetry we may assume that x belongs to the point set of P_1. Now P_1 is incident with at most two faces of G by Lemma 1.6, and hence it is incident with f_2, f_3 and no other face of G. Thus $f = f_2$ or $f = f_3$ as desired. □

We need the following "ear-decomposition" theorem for 2-connected graphs.

Theorem 2.3. *For every* 2*-connected graph* G *there exist subgraphs* $G_0, G_1, \ldots, G_k$ *such that*

(i) G_0 *is a cycle,*

(ii) *for* $i = 1, 2, \ldots, k$ *the graph* G_i *is a path with both ends in* $G_0 \cup G_1 \cdots \cup G_{i-1}$ *and otherwise dijsoint from it, and*

(iii) $G = G_0 \cup G_1 \cdots \cup G_k$.

Theorem 2.4. *The boundary of every face of a* 2*-connected plane graph is the point set of a cycle of* G*.*

Proof. We proceed by induction on $|V(G)|$. If G is a cycle, then the theorem follows from Theorem 1.3. By Theorem 2.3 we may therefore assume that G has a 2-connected subgraph G' such that G is obtained from G' by adding a path P with ends in G' and otherwise disjoint from G'.

Let $f \in F(G)$. Then $f \subseteq f'$ for some face f' of G'. By induction the face f' is bounded by a cycle C of G'. If the interior of P is not a subset of f', then $f = f'$ by Lemma 1.10; thus C is the boundary of f and the theorem holds. We may therefore assume that the interior of P is a subset of f'. Now $\mathrm{bd}(f) \subseteq \overline{f'} \cap G \subseteq C \cup P$, and hence f is a face of $C \cup P$ by Lemma 1.10(ii). Thus f is bounded by a cycle by Lemma 2.2. □

Definition 2.5. We say that a plane graph G is a *straight line plane graph* if each edge of G is a straight line segment. We say that a straight line plane graph G is a *convex plane graph* if every face of G is bounded by a convex polygon. Thus we can speak of a *straight line drawing* and *convex drawing* of a planar graph.

Definition 2.6. We denote the graph obtained from a graph G by contracting the edge e by G/e. A *minor* of a graph G is any graph obtained from a subgraph of G by contracting edges. An H *minor* is a minor isomorphic to H.

Exercise 2.7. (i) If a graph G has an H subdivision, then it has an H minor.

(ii) If H has maximum degree at most three, then a graph G has an H subdivision if and only it has an H minor.

Exercise 2.8. Let H be a fixed graph. Prove that the following statements are equivalent:

(a) For any graph G, the graph G has an H minor if and only if it has an H subdivision.

(b) The graph H has maximum degree at most three.

Lemma 2.9. *Every* 3*-connected graph G on at least five vertices has an edge e such that G/e is* 3*-connected.*

Lemma 2.10. *Every* 3*-connected simple graph with no K_5 or $K_{3,3}$ minor has a convex planar drawing.*

Corollary 2.11. *Every* 3*-connected simple planar graph has a convex planar drawing.*

Our next objective is to replace minors by subdivisions in the statement of Lemma 2.10. The next lemma says that in the context of the presence of K_5 or $K_{3,3}$ the two are actually equivalent.

Lemma 2.12. *A graph G has a K_5 or $K_{3,3}$ minor if and only if it has a K_5 or $K_{3,3}$ subdivision.*

Lemma 2.10 cannot be extended to all graphs, but the weaker conclusion that G has a straight line drawing can. It is possible to deduce that by decomposing graphs into smaller graphs along cutsets of size at most two, but doing so carefully is not very pleasant. It seems preferable to take a different route. The following purely combinatorial lemma will allow us to bypass these technical difficulties.

Lemma 2.13. *Let G be a graph on at least four vertices with no K_5 or $K_{3,3}$ subdivision, and assume that adding an edge joining any pair of nonadjacent vertices creates a K_5 or $K_{3,3}$ subdivision. Then G is* 3*-connected.*

Proof. We proceed by induction on $|V(G)|$. The lemma clearly holds when G has exactly four vertices, and so we may assume that $|V(G)| \geq 5$. We leave proving that G is 2-connected as an exercise. Suppose for a contradiction that $G\backslash u\backslash v$ is disconnected for some two vertices $u, v \in V(G)$. Then G can be written as $G_1 \cup G_2$, where $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$ and $V(G_1) \cap V(G_2) = \{u, v\}$. It follows from the maximality of G that u, v are adjacent in G, and so we may assume that they are adjacent in both G_1 and G_2. By induction each of G_1, G_2 is either 3-connected, or has at most three vertices. By Lemma 2.10 we may assume that G_1, G_2 are plane graphs. For $i \in \{1, 2\}$ let $w_i \in V(G_i)$ be such that u, v, w_i belong to the boundary of some face of G_i.

By hypothesis the graph $G + w_1w_2$ contains a K_5 or $K_{3,3}$ subdivision K. It follows that for some $i \in \{1, 2\}$ all except possibly one branch-vertex of K belong to G_i. From the symmetry we may assume that $i = 1$. Since G_1 has no K_5 or $K_{3,3}$ subdivision and u, v are adjacent in G_1 we deduce that some branch-vertex x of K belongs to G_2. But x is separated from the other branch-vertices of K by the 3-element set $\{u, v, w_1\}$, and hence K is isomorphic to $K_{3,3}$. It also follows that if in G we identify all vertices of $V(G_2) - V(G_1)$ into a single vertex, the new graph is also a counterexample to the lemma. Thus it follows by induction that G is equal to this graph; in other words, G_2 is a triangle with vertices u, v, w_2. But now it follows that $G + w_1w_2$ is planar, contrary to Corollary 2.1. □

Theorem 2.14. *For a graph G the following conditions are equivalent:*

(1) *G is planar,*

(2) *G has a straight line planar drawing,*

(3) *G has no K_5 or $K_{3,3}$ minor,*

(4) *G has no K_5 or $K_{3,3}$ subdivision.*

Proof. Statements (3) and (4) are equivalent by Lemma 2.12. (2) $\Rightarrow$ (1) is trivial and (1) $\Rightarrow$ (4) follows from Corollary 2.1. To prove (3) $\Rightarrow$ (2) let G have no K_5 or $K_{3,3}$ minor. By adding edges to G we may assume that it is edge-maximal in the sense that it satisfies the hypotheses of Lemma 2.13. By that lemma G is 3-connected, and hence it has a straight line drawing by Lemma 2.10. □

Exercise 2.15. Let G be a 3-connected simple graph not isomorphic to K_5. Then G is planar if and only if it has no $K_{3,3}$ subdivision.

3 Uniqueness of planar drawings

Definition 3.1. A cycle C in a graph G is called *peripheral* if it is induced and $G\backslash V(C)$ has at most one component.

Theorem 3.2. *Let G be a 3-connected simple plane graph, and let C be a subgraph of G. Then C bounds a face in G if and only if C is a peripheral cycle.*

Proof. If C is a peripheral cycle, then by Theorem 1.3 one of the faces of C, say f, is disjoint from G, and its boundary is C. By Lemma 1.10 the face f is a face of G, and hence C bounds a face of G, as desired.

Conversely, let C be the boundary of a face $f \in F(G)$. Then C is a cycle of Theorem 2.4. Suppose first that some edge $e \in E(G) - E(C)$ has both its ends, say u, v, on C. Since G is simple the graph $C\backslash u\backslash v$ consists of two components, and since G is 3-connected, there is a path P in $G\backslash u\backslash v$ joining those two components. Let x, y be the ends of P. We may assume that P is otherwise disjoint from C. By adding a new vertex into the face f and joining it to x, y, u, v we obtain a planar drawing of a subdivision of K_5, contrary to Lemma 2.1. This proves that C is induced.

Thus we may assume for a contradiction that $G\backslash V(C)$ has at least two components. Let $a, b \in V(G) - V(C)$ belong to different components of $G\backslash V(C)$. By Menger's theorem there exist three internally disjoint paths P_1, P_2, P_3 between a and b. It follows that each of P_1, P_2, P_3 uses a vertex of C; let $c_i \in V(P_i) \cap V(C)$. The plane graph obtained from G by inserting a new vertex into the face f and joining it to c_1, c_2, c_3 has a $K_{3,3}$ subdivision, contrary to Lemma 2.1. □