WEEK 1 PROBLEMS

Math 6014A

1. Let $n \geq 1$ be an integer. Prove that a sequence of positive integers $d_1, d_2, \ldots, d_n$ is a degree sequence of a tree if and only if $d_1 + d_2 + \cdots + d_n = 2n - 2$.

2. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of points in the plane such that the distance between any two distinct points in S is at least one. Show that there are at most $3n$ pairs of points at distance exactly one.

3. If a graph has minimum degree $\delta \geq 2$, then it contains a cycle of length at least $\delta + 1$.
Hint. Consider a longest path.

4. The *girth* of a graph G is the length of a shortest cycle in G; if G has no cycles we define the girth of G to be infinite. Show that for $k \geq 2$

(a) a k-regular graph of girth 4 has $\geq 2k$ vertices, and up to isomorphism there is exactly one such graph on $2k$ vertices,
(b) a k-regular graph of girth 5 has $\geq k^2 + 1$ vertices,
(c) a k-regular graph of girth 5 in which every two vertices have distance ≤ 2 has exactly $k^2 + 1$ vertices, and find such a graph for $k = 2, 3$. (It is known that such a graph can only exist for $k = 2, 3, 7$, and, possibly, 57.)

Hint. Take a vertex $v \in V(G)$, look at all vertices adjacent to v, and look at all vertices adjacent to these vertices.

5. Let G be a connected graph, and let $r \in V(G)$. Prove that G has a spanning tree T such that for every edge of G with ends u and v, either u belongs to the unique path in T with ends v and r, or v belongs to the unique path in T with ends u and r. (This is called a *depth first search tree* with root r.)

6. Show that if a graph G contains k edge-disjoint spanning trees, then for each partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$, the number of edges of G which have ends in different parts of the partition is at least $k(n-1)$. (Tutte and Nash-Williams have shown that the converse is also true.)