WEEK 11 PROBLEMS

Math 6014

1. (Ramsey's theorem for many colors) Prove that for all integers $n, k, t \geq 1$ there exists an integer N such that for every $c : [\{1, 2, \dots, N\}]^n \to \{1, 2, \dots, t\}$ there exist an integer $i \in \{1, 2, \dots, t\}$ and a set $A \subseteq \{1, 2, \dots, N\}$ of size at least k such that $c(X) = i$ for every $X \in [A]^n$.

2. (Infinite Ramsey's theorem) Let $\mathbf{N}$ denote the set of all positive integers. Prove that for all integers $n, t \geq 1$ and for every $c : [\mathbf{N}]^n \to \{1, 2, \dots, t\}$ there exist an integer $i \in \{1, 2, \dots, t\}$ and an infinite set $A \subseteq \mathbf{N}$ such that $c(X) = i$ for every $X \in [A]^n$.

3. Let $n \geq 1$ be a natural number, and let G_n be the graph with vertex-set all 2-element subsets of $\{1, 2, \dots, n\}$, where for $a < b < c$ the vertex $\{a, b\}$ is adjacent to $\{b, c\}$, and those are all the edges of G_n. Prove that G_n has no triangle, and that $\chi(G_n) \to \infty$ as $n \to \infty$.

4. Let $s, t \geq 2$ be integers, and let T be a tree on t vertices. Prove that if G is a graph on $(s - 1)(t - 1) + 1$ vertices, then either G has a K_s subgraph, or the complement of G has a subgraph isomorphic to T. Prove that the number $(s - 1)(t - 1) + 1$ is best possible.

5. Prove that $\alpha(G \boxtimes H) \leq r_2(\alpha(G) + 1, \alpha(H) + 1) - 1$. Deduce that $\alpha(C_5 \boxtimes C_5) \leq 5$.

6. Prove that every sequence of $(k - 1)(l - 1) + 1$ real numbers has either a non-decreasing subsequence of length k or a non-increasing subsequence of length l.

7. (Schur's theorem) Prove that for every integer $t \geq 1$ there exists an integer N such that for every $c : \{1, 2, \dots, N\} \to \{1, 2, \dots, t\}$ there exist integers $x, y, z \in \{1, 2, \dots, N\}$ such that $c(x) = c(y) = c(z)$ and $x + y = z$.
Hint. Consider a (not necessarily proper) coloring of the edges of the complete graph with vertex-set $\{1, 2, \dots, N\}$, where the color of the edge ij depends only on $|i - j|$.