WEEK 14 PROBLEMS

Math 6014A

1. Generalize the crossing number lemma to multigraphs, as follows. For every integer m there exists a constant $c > 0$ such that if G is a loopless multigraph with at least $4m|V(G)|$ edges and every two vertices of G are joined by at most m edges, then the crossing number of G is at least $c|E(G)|^3/|V(G)|^2$.

2. Let k, n be integers satisfying $2 \leq k \leq \sqrt{n}$. Given a set of n points in the plane, the number of lines containing at least k of them is at most cn^2/k^3, where c is an absolute constant.
Hint. Use the proof of the Szemerédi-Trotter theorem, but use only lines containing at least k points.

3. Prove that there exists an absolute constant c such that for every set of n points in the plane the number of pairs of those points that are at distance one is at most $cn^{4/3}$.
Hint. Define a graph as follows. The vertices are the given points. Draw a unit circle around each point. For each circle, consecutive points on the circle will be joined by an edge of the graph. Apply the crossing number lemmma, but beware of loops and parallel edges.

4. Prove that there exists an absolute constant such that the following holds. Given n points in the plane such that no 2015 of them are on a line, the number of distinct distances they determine is at least cn.
Hint. Let D be the set of distinct distances. Define a graph as follows. The vertices are the given points. Draw $|D|n$ circles around each point, for each $r \in D$ there will be a circle of radius r, and define edges similarly as in the previous problem.

5. Let $k \geq 1$ be an integer, let $P = \{(i, j) : i = 0, 1, \dots, k-1, j = 0, 1, \dots, 4k^2 - 1\}$, and let $\mathcal{L}$ be the set of lines with equations $y = ax + b$, where $a = 0, 1, \dots, 2k-1$ and $b = 0, 1, \dots, 2k^2 - 1$. Prove that for each $i = 0, 1, \dots, k-1$ each line in $\mathcal{L}$ contains a point in P with x-coordinate equal to i. Deduce that the bound in the Szemerédi-Trotter theorem is asymptotically tight when $n = m$.

6. Prove that the bound in the Szemerédi-Trotter theorem is asymptotically tight whenever $n^2 \leq m$ and $m^2 \leq n$.