# WEEK 3 PROBLEMS

Math 6014A

**1.** Let $T_1$ and $T_2$ be two spanning trees of a connected graph $G$. Prove that $T_1$ can be transformed to $T_2$ through a sequence of intermediate trees, each arising from the previous one by deleting an edge and adding another one.

**2.** Let $G$ be a graph, let $A \subseteq V(G)$, let $u \in V(G) - A$, and let $k$ be an integer. Prove that either
(i) there are $k$ paths, each between $u$ and some vertex of $A$ with only $u$ in common, or
(ii) there is a set $X \subseteq V(G) - \{u\}$ with $|X| < k$ such that $G \backslash X$ has no path between $u$ and $A$,
and not both.

**3.** A *circulation* in a directed graph $D$ is a function $g : E(D) \to \mathbf{R}$ satisfying the conservation condition at every vertex. Let $l, u : E(D) \to \mathbf{R}_0^+$ be a *lower capacity function* and *upper capacity function*, respectively, and assume that $l(e) \leq u(e)$ for every edge $e \in E(D)$. A circulation $g$ is *feasible* if $l(e) \leq g(e) \leq u(e)$ for every edge $e \in E(D)$. Prove that there exists a feasible circulation if and only if $l^+(A) \leq u^-(A)$ for every set $A \subseteq V(D)$.
*Hint.* Add a vertex $s$ and an edge from $s$ to every vertex of $D$, and a vertex $t$ and an edge from every vertex of $D$ to $t$. Define $c(sv) = l^-(v)$, $c(vt) = l^+(v)$, and $c(e) = u(e) - l(e)$ for $v \in V(D)$ and $e \in E(D)$. Given a flow $f$ of value $\sum_{e \in E(D)} l(e)$ in the network thus defined, consider $f + l$.

**4.** Let $f$ be a flow in a network $N = (D, c, s, t)$, and let $f'$ be obtained from a shortest $f$-augmenting path as in the proof of the Max-Flow Min-Cut theorem. Define a digraph $D_f$ by saying that $V(D_f) = V(D)$ and $\overrightarrow{uv} \in E(D_f)$ if either $\overrightarrow{uv} \in E(D)$ and $f(\overrightarrow{uv}) < c(\overrightarrow{uv})$, or $\overrightarrow{vu} \in E(D)$ and $f(\overrightarrow{vu}) > 0$. Prove that for every $v \in V(D)$ we have $\mathrm{dist}_{D_{f'}}(s, v) \geq \mathrm{dist}_{D_f}(s, v)$ and $\mathrm{dist}_{D_{f'}}(v, t) \geq \mathrm{dist}_{D_f}(v, t)$.

**5.** Let $N, f$ and $f'$ be as in the previous problem, and assume that the shortest $f$-augmenting path has length $k$. Let $E_f$ denote the set of all edges that belong to an $f$-augmenting path of length $k$. Prove that $E_{f'}$ is a proper subset of $E_f$.

**6.** Deduce that by choosing an augmenting path of minimum length at every step, the process from the proof of the Max-Flow Min-Cut theorem will result in a maximum flow after at most $|V(D)| \cdot |E(D)|$ augmentations.