WEEK 5 PROBLEMS

Math 6014A

1. Prove that every regular bipartite multigraph has a perfect matching.

2. Two people play a game on a graph G by alternately selecting distinct vertices $v_0, v_1, \ldots$ such that, for $i > 0$, v_i is adjacent to v_{i-1}. The last player able to select a vertex wins. Show that the first player has a winning strategy if and only if G has no perfect matching.
Hint. Augmenting paths.

3. A *vertex cover* of a graph G is a set of vertices A such that every edge has at least one end in A. Prove that in a bipartite graph the size of a maximum matching is equal to the size of a minimum vertex cover. Prove that this does not hold for all graphs.

4. (Petersen's theorem) Prove that every 2-edge-connected 3-regular graph has a perfect matching.
Hint. Use Tutte's theorem.

5. Let $A_1, A_2, \ldots, A_m$ be subsets of a set S. A *system of distinct representatives* for the family $(A_1, A_2, \ldots, A_m)$ is a subset $\{a_1, a_2, \ldots, a_m\}$ of S such that $a_i \in A_i$ for $i = 1, 2, \ldots, m$, and $a_i \neq a_j$ for $i \neq j$. Show that $(A_1, A_2, \ldots, A_m)$ has a system of distinct representatives if and only if $\left|\bigcup_{i \in J} A_i\right| \geq |J|$ for all subsets J of $\{1, 2, \ldots, m\}$.

6. Let G be a bipartite graph with bipartition (A, B) such that
$(*)$ $|N(S)| \geq |S|$ for every $S \subseteq A$.
Assume furthermore that $E(G)$ is minimal subject to $(*)$; that is, $G \backslash e$ does not satisfy $(*)$ for every edge $e \in E(G)$. Prove, without using Hall's theorem, that $E(G)$ is a matching of size $|A|$. Deduce Hall's theorem.