**WEEK 6 PROBLEMS**

Math 6014A

**1.** Let $n \geq 1$ be an integer. Prove that the maximum number of subsets of $\{1, 2, \ldots, n\}$ such that none is a subset of another is $\binom{n}{\lfloor n/2 \rfloor}$. (This is called Sperner's lemma.)
*Hint*. Find a cover by $\binom{n}{\lfloor n/2 \rfloor}$ chains.

**2.** An $r \times s$ *Latin rectangle* based on $1, 2, \ldots, n$ is an $r \times s$ matrix $A = (a_{ij})$ such that each entry is one of the integers $1, 2, \ldots, n$ and each integer occurs in each row and column at most once. Prove that every $r \times n$ Latin rectangle $A$ can be extended to an $n \times n$ *Latin square*. [*Hint.* Assume that $r < n$ and extend $A$ to an $(r+1) \times n$ Latin rectangle. Let $A_j$ be the set of possible values $a_{r+1,j}$, that is let $A_j = \{k : 1 \leq k \leq n, k \neq a_{ij}\}$. Check that $\{A_j : 1 \leq j \leq n\}$ has a set of distinct representatives.] Prove that there are at least $n!(n-1)! \cdots (n-r+1)!$ distinct $r+n$ Latin rectangles based on $1, 2, \ldots, n$.

**3.** Let $A$ be an $r \times s$ Latin rectangle based on $1, 2, \ldots, n$ and denote by $A(i)$ the number of times the symbol $i$ occurs in $A$. Show that $A$ can be extended to an $n \times n$ Latin square if and only if $A(i) \geq r + s - n$ for every $i = 1, 2, \ldots, n$.

**4.** (Dual of Dilworth's Theorem) Let $(P, \leq)$ be a partially ordered set such that every chain has at most $k$ elements. Prove that $P$ can be decomposed into $k$ antichains.

**5.** Prove that the edge-chromatic number of a bipartite multigraph $G$ is $\Delta(G)$.

**6.** A set $F \subseteq E(G)$ is called an *edge cover* if every vertex of $G$ is incident with at least one edge in $F$. Let $G$ be a bipartite graph with minimum degree at least one. Prove that the size of a maximum independent set is equal to the size of a minimum edge cover.

**7.** Let $k \geq 0$ be an integer, and let $G$ be a graph. Prove that $G$ has a matching saturating all but at most $k$ vertices of $G$ if and only if $o(G \backslash X) \leq |X| + k$ for every $X \subseteq V(G)$.

**8.** Design a polynomial-time algorithm with the following specifications:
*Input:* A graph $G$ and a matching $M$ in $G$ saturating all except exactly $k$ vertices of $G$.
*Output:* Either a matching in $G$ of size at least $|M| + 1$, or a set $X \subseteq V(G)$ such that $o(G \backslash X) = |X| + k$ (in which case $G$ has no matching of size $|M| + 1$ by the previous exercise).
Use your algorithm to design a polynomial-time algorithm to find a maximum matching in a graph.