WEEK 7 PROBLEMS

Math 6014A

1. Prove that bipartite graphs and their complements are perfect. Do not use the weak or strong perfect graph theorem.

2. Prove that line graphs of bipartite graphs and their complements are perfect. Do not use the weak or strong perfect graph theorem.

3. A graph G is a *comparability graph* if there exists a poset $(P, \leq)$ such that $V(G) = P$ and two distinct vertices $u, v \in V(G)$ are adjacent if $u \leq v$ or $v \leq u$. Prove that comparability graphs and their complements are perfect. Do not use the weak or strong perfect graph theorem.

4. For an integer $k \geq 0$ let $p_G(k)$ be the number of proper colorings of a multigraph G using k colors. Prove that for every non-loop edge e of G and every integer $k \geq 0$,

$$p_G(k) = p_{G\backslash e}(k) - p_{G/e}(k).$$

Deduce that $p_G(k)$ is a polynomial in k of degree $n := |V(G)|$. It is called the *chromatic polynomial.* Determine the coefficients of k^n and k^{n-1}.

5. Let G be a connected simple graph. Prove that $p_G(k) \leq k(k-1)^{|V(G)|-1}$ for all integers $k \geq 0$. Prove that equality holds for all integers $k \geq 0$ if and only if G is a tree.

6. For every positive integer k construct a simple graph G with $\chi(G) \geq k$ and no triangles.
Hint. If G_k works for k, construct G_{k+1} as follows. Let $V(G_k) = \{v_1, v_2, \ldots, v_n\}$, add $n+1$ new vertices $\{u, u_1, u_2, \ldots, u_n\}$, and for $i = 1, 2, \ldots, n$ join u_i to the neighbors of v_i and to u.

7. A graph G is *chordal* if G has no induced cycle of length at least four. In other words, every cycle of length at least four has a chord (an edge with both ends on the cycle, which does not belong to the cycle). Let G be a chordal graph.

(a) Let $X \subseteq V(G)$ be a minimal set such that $G\backslash X$ is disconnected. Prove that X is a clique.

(b) Prove that G has a *simplicial* vertex; that is, a vertex whose set of neighbors is a clique.

(c) Prove that G and its complement are perfect. Do not use the weak or strong perfect graph theorem.

8. A graph is *P_4-free* if it has no induced subgraph isomorphic to a path on four vertices.

(a) Prove that a graph G is P_4-free if and only if every induced subgraph H of G has the property that every maximal clique in H intersects every maximal independent set in H.

(b) Prove that P_4-free graphs are perfect. Do not use the weak or strong perfect graph theorem.