WEEK 8 PROBLEMS

Math 6014A

1. Let $x_1, x_2, \ldots, x_n$ be vectors of norm at least 1 in a Euclidean space. Prove that there are at most $\lfloor n^2/4 \rfloor$ unordered pairs i, j such that $1 \le i, j \le n$ and $|x_i + x_j| < 1$.

2. Let $\{x_1, x_2, \ldots, x_n\}$ be a set of diameter 1 in the plane. Prove that the maximum possible number of pairs of points at distance greater than $1/\sqrt{2}$ is $\lfloor n^2/3 \rfloor$.

3. Let G be a graph on n vertices which does not contain a subgraph isomorphic to a cycle with a diagonal. (In other words, every cycle in G is an induced subgraph.) Prove that if $n \ge 4$ then $|E(G)| \le 2n - 4$.
Hint. Reduce to 2-connected graphs.

4. Let G be a graph on at least two vertices with $|E(G)| \ge 2|V(G)| - 2$. Prove that G has a subgraph isomorphic to a subdivision of K_4.

5. Let G be a triangle-free graph on n vertices. Prove that $\sum_{v \in V(G)} (\deg(v))^2 \le n^3/4$.

6. Let G be a connected graph on n vertices such that $\deg(u) + \deg(v) \ge n$ for every two non–adjacent vertices $u, v \in V(G)$. Prove that G is Hamiltonian.
Suggestion. Take a longest path P and by studying the endpoints of P show that there is a cycle with $V(C) = V(P)$. Show that $V(C) = V(G)$.