CHAPTER 1

Series-Parallel Graphs

Series-parallel graphs are a useful class of graphs—they are fairly simple and reasonably well-understood to allow easy proofs for many results, and at the same time they are rich enough so that many problems are non-trivial even when restricted to this class. In particular, series-parallel graphs are a fertile testing ground for various conjectures. We will use series-parallel graph as a first example of graph structure and the corresponding decomposition.

1.1. Preliminaries

Graphs are undirected, may have loops and multiple edges, and are finite, unless stated otherwise. More precisely, a *graph* G consists of a set of vertices $V(G)$, a set of edges $E(G)$ and a set of incidences between vertices and edges. Every edge is incident with two (not necessarily distinct) vertices, called its *ends*. If its ends are equal it is called a *loop*, otherwise it is called a *link*. Two edges are *parallel* if they have the same ends. A graph is *simple* if it has no loops and no parallel edges. *Paths* and *cycles* have no repeated vertices or edges, and are regarded as graphs. The graph with no vertices is called the *null graph* and is denoted by K_0. The null graph is (by definition) neither connected nor disconnected, and has no components. For an integer $n \geq 1$ we denote by K_n the complete graph on n vertices. If G is a graph and X is a vertex, a set of vertices, an edge, or a set of edges, then $G\backslash X$ denotes the graph that results when X is deleted from G. If G_1, G_2 are graphs then $G_1 \cup G_2$ is the graph with vertex-set $V(G_1) \cup V(G_2)$, edge-set $E(G_1) \cup E(G_2)$ and the obvious incidences. The graph $G_1 \cap G_2$ is defined similarly. A *separation* in a graph is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$. The *order* of (A, B) is $|A \cap B|$. If G is a graph and $X \subseteq V(G)$ we denote by $\delta(X)$ the set of edges with one end in X and the other in $V(G) - X$. A set $E \subseteq E(G)$ is called a *cut* if $E \neq \emptyset$ and $E = \delta(X)$ for some $X \subseteq V(G)$. (This is not the same as saying that $G\backslash E$ is disconnected.) An inclusion-wise minimal cut is called a *bond* of G. It is easy to see that every cut is a disjoint union of bonds. If v is a vertex of a graph, then the *degree*, denoted by $d_G(v)$ or $d(v)$, is the number of edges of G incident with v, counting loops twice. A *tree* is a connected graph with no cycles. A *forest* is a graph with no cycles. We say that two paths P, Q are *internally disjoint* if every vertex common to P and Q is an end of both.

Exercise 1.1.1. A cut $\delta(X)$ in a connected graph G is a bond if and only if $G\backslash X$ and $G\backslash(V(G) - X)$ are both connected.

Exercise 1.1.2. Let $T_1, T_2, \ldots, T_k$ be subtrees of a tree such that every two of them have a vertex in common. Prove that they all have a vertex in common.

Exercise 1.1.3. Prove that if we choose a direction for every edge of a tree T, then for some vertex t of T all the edges incident with t will be directed toward t.

1.2. Subdivisions and minors

In this section we define subdivisions and minors of graphs, and state a few straightforward exercises to illustrate these concepts.

Definition 1.2.1. Let G be a graph, and let $e \in E(G)$. As explained earlier, $G\backslash e$ denotes the graph obtained from G by *deleting* e. By G/e we denote the graph obtained from G by *contracting* e, where contraction is defined as follows. If e is a loop then G/e is defined to be $G\backslash e$; otherwise G/e is obtained by deleting e and identifying the ends of e. Thus $|E(G/e)| = |E(G\backslash e)| = |E(G)| - 1$, and if e is a link then $|V(G/e)| = |V(G)| - 1$. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by repeatedly contracting edges. We say that G has an H *minor* if G has a minor isomorphic to H.

Definition 1.2.2. Let G, H be graphs, and let $e \in E(H)$. We say that G is obtained from H by *subdividing* e if G is obtained from H by deleting e, adding a new vertex $w \notin V(H)$ and joining w to the ends of e by two new edges. Thus $|V(G)| = |V(H)|+1$ and $|E(G)| = |E(H)|+1$. The reverse operation, which (up to isomorphism) consists of contracting an edge incident with w, is called *supressing* w. We say that G is a *subdivision* of H if G is obtained from H by repeatedly applying the operation of subdividing and edge. We say that H *is topologically contained in* G, or that G *topologically contains* H or that G *has an* H *subdivision* if G has a subgraph isomorphic to a subdivision of H. Thus if G has an H subdivision, then it has an H minor.

Exercise 1.2.3. The statement "G has an H subdivision if and only if it has an H minor" holds for every G if and only if H has maximum degree at most three.

Exercise 1.2.4. A graph G has an H subdivision if and only if a graph isomorphic to H can be obtained from G by applying the following operations:

(i) deleting an edge,
(ii) supressing a vertex of degree two,
(iii) deleting an isolated vertex.

Exercise 1.2.5. A graph G has an H minor if and only if a graph isomorphic to H can be obtained from G by applying the following operations in arbitrary order:

(i) deleting an edge,
(ii) contracting an edge,
(iii) deleting an isolated vertex.

Exercise 1.2.6. A graph G has an H subdivision if and only if there is a 1–1 mapping $\phi : V(H) \to V(G)$ and for every $e \in E(H)$ there is a subgraph P_e of G such that

(i) if $e \in E(H)$ has distinct ends u, v, then P_e has ends $\phi(u), \phi(v)$,
(ii) if $e \in E(H)$ is a loop incident with $u \in V(H)$, then P_e is a cycle containing $\phi(u)$, regarded as a closed path,
(iii) for distinct $e, e' \in E(H)$ the paths $P_e, P_{e'}$ are internally disjoint.

Exercise 1.2.7. A graph G has an H minor if and only if for every $v \in V(H)$ there exists a connected subgraph G_v of G and a mapping $\psi : E(H) \to E(G)$ such that

(i) for distinct $u, v \in V(H)$, G_u and G_v are vertex-disjoint,
(ii) ψ is 1–1 and $\psi(e) \notin E(G_v)$ for every $e \in E(H)$ and $v \in V(H)$,

(iii) if $e \in E(H)$ has ends u, v, then $\psi(e)$ has one end in G_u and the other in G_v.

Exercise 1.2.8. Let H be a graph of maximum degree at most three. Then a graph G has an H subdivision if and only if it has an H minor.

Exercise 1.2.9. Characterize all graphs H such that for every graph G the graph G has an H subdivision if and only if it has an H minor.

1.3. Characterization and recognition of series-parallel graphs

Definition 1.3.1. A graph is *series-parallel* if it has no K_4 subdivison.

Exercise 1.3.2. No series-parallel graph is 3-connected.

Exercise 1.3.3. Every simple series-parallel graph has a vertex of degree at most two.

Definition 1.3.4. Let G be a graph. By a *series-parallel reduction* we mean any of the following operations:

(i) deletion of a loop,
(ii) deletion of a vertex of degree at most 1,
(iii) deletion of a parallel edge,
(iv) suppression a vertex of degree two.

We can now state the main characterization of series-parallel graphs, which will also explain where the name comes from.

Theorem 1.3.5. *A graph is series-parallel if and only if it can be reduced to the null graph by repeatedly applying the series-parallel reductions.*

Exercise 1.3.6. Prove Theorem 1.3.5.

Theorem 1.3.5 may be used to design a linear-time algorithm to test if an input graph is series-parallel. We leave this as an exercise, but it should be noted that the algorithm is less trivial than it may seem. How do you make sure that the total time spent on deleting parallel edges is at most linear?

Exercise 1.3.7. Find a linear-time algorithm to test whether an input graph is series-parallel.

Definition 1.3.8. A graph is *chordal* if it has no **induced** cycle of length four or more.

Definition 1.3.9. Let $k \geq 0$ be an integer, let G_1, G_2 be graphs on disjoint vertex-sets, and let A_i be a clique of size k in G_i. Let G be obtained from $G_1 \cup G_2$ by first identifying A_1 and A_2 into a set A, and then deleting an arbitrary subset of edges whose ends are distinct vertices of A. In those circumstances we say that G is a *k-sum* of G_1 and G_2. We also say that G is a *clique-sum* and a *$\leq t$-sum* of G_1 and G_2 for all $t \geq k$. Thus the result of this operation is not unique.

We end this section with three other characterizations of series-parallel graphs stated in the form of exericses.

Exercise 1.3.10. A graph is series-parallel if and only if it can be obtained by repeated $\leq$ 2-sums starting from graphs on at most three vertices.

Exercise 1.3.11. A graph is series-parallel if and only if it can be obtained by repeated clique-sums starting from graphs on at most three vertices.

Exercise 1.3.12. A graph G is series-parallel if and only if it is a subgraph of a chordal graph with no K_4 subgraph.

Exercise 1.3.13. A 2-*terminal graph* is a graph G with two specified distinct vertices s and t, called the *source* and *sink*, respectively. A *series composition* of two 2-terminal graphs (G_1, s_1, t_1) and (G_2, s_2, t_2) is the 2-terminal graph with source s_1 and sink t_2 obtained from the disjoint union of G_1 and G_2 by identifying t_1 and s_2. The *parallel composition* of the same 2-terminal graph is obtained by identifying their sources and identifying their sinks to form the source and sink of the resulting 2-terminal graph. Prove that a 2-terminal graph G can be obtained from the unique 2-terminal graph (G, s, t) with two vertices and one edge by repeatedly applying the series and parallel compositions if and only if G is series-parallel and the graph $G + st$ has no cut vertex.