Graph Algorithm Note: Apr 08, 2009

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1 Homework 2

Find an explicit constant c and an algorithm to find a maximum independent set in a planar graph in time $O(2^{c\sqrt{n}})$, where n is the number of vertices of the graph. (You can use the material from course website.) Due by 4/15/2009.

2 An application: approximating $\alpha(G)$ for planar graphs

We recall the Separator Theorem.

Theorem 2.1 *For any planar graph G on n vertices, there exists a partition (A, B, C) of $V(G)$ such that*

(i) $|C| \leq 2\sqrt{2}\sqrt{n}$;

(ii) $|A \cup C|, |B \cup C| > \frac{n}{3}$;

(iii) there is no edge from A to B.

Lemma 2.2 *Let G be a planar graph on n vertices, and let $\varepsilon \in (0, 1)$. Then there is a set $X \subset V(G)$ of size $O(\sqrt{\frac{n}{\varepsilon}})$ such that no component of $G - X$ has more than εn vertices.*

Proof. Let $X := \emptyset$. If some component H of $G - X$ has no less than εn vertices, then apply the Separator Theorem to H to get a partition (A, B, C) as in the theorem: $|C| \leq \sqrt{8n}, |A| \leq \frac{2}{3}|V(H)|, |B| \leq \frac{2}{3}|V(H)|$. Set $X := X \cup C$, repeat.

We classify the components arising from in the algorithm into levels: level 0 component is one we will end up with; Level i component is one such that after applying the algorithm to it, all resulting components are of level $j \leq i - 1$, and at least one is of level exactly $i - 1$.

Each level i component has more than $(\frac{3}{2})^{i-1}\varepsilon n$ vertices, for $i \geq 1$. Then, the number of level i components is no more than $\frac{n}{(\frac{3}{2})^{i-1}\varepsilon n} = (\frac{2}{3})^{i-1}/\varepsilon$.

We may assume that $\varepsilon \geq \frac{1}{n}$; otherwise, $\frac{1}{\varepsilon} \geq n$, and so $\sqrt{\frac{n}{\varepsilon}} \geq n$, thus $X = V(G)$ satisfies the conclusion of lemma, because $|X| = n = O(\sqrt{\frac{n}{\varepsilon}})$. Let there be k levels. Then, $1 \leq (\frac{2}{3})^{k-1}/\varepsilon \leq (\frac{2}{3})^{k-1}n$, then $(\frac{3}{2})^{k-1} \leq n$, then $k \leq \log_{3/2} n + 1$.

Fix level i, and let $L_1, L_2, ..., L_t$ be the components of level i. Let $n_j = |V(L_j)|$. How many vertices get added to X at this level? Since $n_1 + n_2 + ... + n_t = Const \leq n$,

$$\sum_{j=1}^{t} \sqrt{8n_j} \leq \sqrt{8}t\sqrt{\frac{n}{t}} \leq \sqrt{8nt} \leq \sqrt{8}\sqrt{n}(\frac{2}{3})^{(i-1)/2}/\sqrt{\varepsilon}.$$

Thus, the total size of X is at most

$$O(\sqrt{\frac{n}{\varepsilon}})\sum_{i\geq 1}(\frac{2}{3})^{(i-1)/2} = O(\sqrt{\frac{n}{\varepsilon}}).$$

Thus this proves the lemma. ∎

Lemma 2.3 *The set X can be found in time $O(n\log n)$, assuming a linear-time separator algorithm.*

Proof. Use the above algorithm. There are $O(\log n)$ levels, each take linear time. ∎

Algorithm 1 (approximating $\alpha(G)$ for planar graphs): Pick $\varepsilon := \frac{\log n}{n}$. We may find a set X as in the Lemma in time $O(n\log n)$. For each component H of $G - X$, we find $\alpha(H)$ exactly by looking at all subsets of $V(H)$ in time $O(2^{|V(H)|}) = O(n)$. (So the running time is $O(n^2)$.)

Let I be the union of the maximum independent sets in H (over all components H of $G-X$), and let I_{opt} be the maximum independent set in G. Then,

$$\frac{|I_{opt}| - |I|}{|I_{opt}|} \leq \frac{|X|}{|I_{opt}|} \leq \frac{n/\sqrt{\log n}}{n/5} = O(\frac{1}{\sqrt{\log n}}).$$

Here, $|I_{opt}| \geq \frac{n}{5}$ since any planar graph is 5-choosable. And the running time is $O(n^2)$.

Algorithm 2 (approximating $\alpha(G)$ for planar graphs): Pick $\varepsilon := \frac{\log\log n}{n}$. We may find a set X as in the Lemma in time $O(n\log n)$. For each component H of $G - X$, we find $\alpha(H)$ exactly by looking at all subsets of $V(H)$ in time $O(2^{|V(H)|}) = O(\log n)$.

Then,

$$\frac{|I_{opt}| - |I|}{|I_{opt}|} \leq \frac{|X|}{|I_{opt}|} \leq \frac{n/\sqrt{\log\log n}}{n/5} = O(\frac{1}{\sqrt{\log\log n}}).$$

And the running time is $O(n\log n)$.

3 Matrix Decomposition

Considering

$$A\overline{X} = \overline{b},$$

where A is a symmetric positive definite. Let G be the corresponding graph: $i \sim j \Leftrightarrow a_{ij} \neq 0, i \neq j, V(G) = \{1, ..., n\}$.

Assume that G is planar.

A can be wrote as $A = LDL^t$, where L is lower-triangular, D is diagonal. $L\overline{y} = \overline{b}$, $D\overline{Z} = \overline{y}$, $L^t\overline{X} = \overline{Z}$.

Objective is to reorder $V(G)$ (rows and columns of A; replace A by PAP^t for some permutation matrix P) to get the "fill-in" under control (to minimize the number of "fill-in"), where "fill-in" refers to non-zero entries of L with corresponding entry of A zero.

Theorem 3.1 *If A is symmetric positive definite, G is a planar on n vertices, then there is a permutation matrix P such that the number of the fill-in of the matrix PAP^t is $O(n\log n)$.*

Theorem 3.2 *There is a permutation matrix P such that the factorization $PAP^t = LDL^t$ requires $O(n^{3/2})$ multiplication.*