

Havens and Tree Width

Scribe: Rishi Saket, March 14, 2005

In the previous lecture we defined a haven of order $k$. In this lecture we will prove a relationship between the order of a haven and tree width of the graph.

A **haven of order** $k$ is a map $\beta : [V(G)]^k \rightarrow 2^V(G)$, such that $\beta(X)$ is the vertex set of a component of $G \setminus X$. Also, $X \subseteq Y \Rightarrow \beta(X) \supseteq \beta(Y)$.

**Theorem 0.1** $\text{tw}(G) \leq k \iff G$ has no haven of order $k + 1$.

**Proof:** We will only prove the forward implication which is easy to see. We know as a fact from the previous lecture that a haven of order $k + 1$ gives an escape strategy for the robber against $k$ cops. Also, if the tree width is at most $k$, then no such escape strategy exists for $k$ cops. \[\Box\]

The reverse implication is harder to prove and instead we will prove a weaker statement.

**Theorem 0.2** If $\text{tw}(G) \geq 3k - 1$, then $G$ has a haven of order $k$.

**Proof:** Let $(T, W)$ be a tree decomposition of $G$ such that,

1. $|W_t| \leq 3k - 1$, $\forall t \in V(T)$ of degree $> 1$.

2. $|W_t \cap W_{t'}| \leq 2k - 1$, $\forall \{t, t'\} \in E(T)$.

3. Number of vertices of $G$ that only belong to $W_t$ with $|W_t| \geq 3k$ is minimized.

Clearly such tree decomposition exists since we can construct a trivial single edge tree with one vertex containing all the vertices, and it satisfies [1] and [2].

We know that the tree width of $(T, W)$ is at least $3k - 1$. This means that there is a leaf $t \in V(T)$ with $|W_t| \geq 3k$. Let $t'$ be the unique neighbor of $t$ in $T$. From our construction we know $|W_t \cap W_{t'}| \leq 2k - 1$.

Let $H := G[W_t]$, that is, $H$ is the subgraph induced by $W_t$. Now suppose that there exists a $Z \subseteq V(H)$ with $|Z| < k$ such that $|V(J) \cap W_t \cap W_{t'}| < k$, $\forall$ components $J$ of $H \setminus Z$. 


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Let $T'$ be obtained from $T$, by adding new leaf vertices $t_J$ for every component $J$ of $H \setminus Z$, all adjacent to $t$. Define the bags $W'_r$ for $T'$ as follows,

$$W'_r := \begin{cases} W_r & \text{if } r \in V(T) - t \\ (W_t \cap W'_r) \cup Z & \text{if } r = t \\ V(J) \cup Z & \text{if } r = t_J \text{ for } J \in \text{comp}(H \setminus Z) \end{cases} \tag{1}$$

Using the fact that $|W_t \cap W'_r| \leq 2k - 1$, $|Z| < k$ and $|V(J) \cap W_t \cap W'_r| < k, \forall J \in \text{comp}(H \setminus Z)$, we get that $|W'_r| \leq 3k - 2$ and $|W'_r \cap W'_r| \leq 2k - 2$, $\forall J \in \text{comp}(H \setminus Z)$. In both cases we have a slack since we are allowed upto $3k - 1$ and $2k - 1$ instead of $3k - 2$ and $2k - 2$ respectively from [1] and [2]. Therefore, $Y$ can be chosen such that $Y - (W_t \cap W'_r) \neq \phi$. This will lead to a contradiction to [3], since the number of vertices of $G$ that only belong to bags of width at least $3k$ is decreased by $|Y - (W_t \cap W'_r)|$.

So we may assume that no such $Z$ exists. Thus $\forall Z \subseteq V(H)$, with $|Z| < k$, some component $J$ of $H \setminus Z$, satisfies $|V(J) \cap W_t \cap W'_r| \geq k$. Define $\beta(Z) := V(J)$. We claim that $\beta$ is a haven of $H$ of order $k$. For this we must check that if $Z \subseteq Z' \subseteq [V(H)]^{<k}$, then $\beta(Z) \supseteq \beta(Z') \in V(J)$. If not, then $\beta(Z) \cap \beta(Z') = \phi$. But we have $|\beta(Z) \cap W_t \cap W'_r| \geq k$ and $|\beta(Z') \cap W_t \cap W'_r| \geq k$, which is a contradiction to $|W_t \cap W'_r| \leq 2k - 1$. Therefore, $H$ has a haven of order $k$, and since $H$ is a subraph of $G$, $G$ has a haven of order $k$.

**Example.** A $K_t$ minor in $G$ gives rise to a haven of order $t$.

**Proof.** Since $G$ has a $K_t$ minor, it has disjoint connected subgraphs $J_1, J_2, \ldots, J_t$ such that $\forall 1 \leq i < j \leq t$, there is a $J_i - J_j$ edge. Let $X \in [V(G)]^{<t}$, define $\beta(X)$ to be the component of $G \setminus X$ that includes some $J_i$, since there is at least one $J_i$ such that $J_i \cap X = \phi$. This is well defined because there cannot be two components of $G \setminus X$ one containing $J_i$ and the other containing $J_j$, since there is a $J_i - J_j$ edge.

**Example.** A $t \times t$ grid minor also defines a haven of order $t$.

**No Proof.**
Definition. Let \( \beta \) be a haven of order \( k \) in a graph \( G \). We say that a set \( X \in [V(G)]^k \) is free if there is no set \( Y \subseteq X \cup \beta(X) \) with \( |Y| < |X| \) such that \( \beta(Y) \subseteq \beta(X) \).

Lemma 0.3 Let \( \beta \) be a haven of order \( k \) in a graph \( G \), and let \( X \subseteq V(G) \) be a free set. Then, for every two disjoint sets \( A, B \subseteq X \), with \( |A| = |B| \), there exists \( |A| \) disjoint \( A - B \) paths in \( H := G[A \cup B \cup \beta(X)] \).

Proof. Suppose not; then \( \exists Z \subseteq V(H) \), such that \( |Z| < |A| \), such that no component of \( H \setminus Z \) intersects both \( A \) and \( B \) (Menger’s Theorem). Notice that \( \beta(X \cup Z) \) is well defined and is a subset of \( \beta(X) \). By the definition of \( Z \), \( \beta(X \cup Z) \) has no neighbor in one of \( A \) or \( B \), say in \( B \). But that means that \( \beta(X \cup Z) = \beta(X \cup Z - B) \subseteq \beta(X) \). Since \( (X \cup Z - B) \subseteq X \cup \beta(X) \) and \( |X \cup Z - B| < |X| \), this is a contradiction to the freedom of \( X \).

Lemma 0.4 Let \( \beta \) be a haven of order \( k \). Then \( \forall t < k \), there exists a free set of size \( t \).

Proof. Let \( X \in V(G) \) such that,

a. \( |X| = t \).

b. subject to a. \( \beta(X) \) is minimal.

Now, \( X \) is a free set. Suppose not, then there is a set \( Y \subseteq X \cup \beta(X) \), such that \( |Y| < |X| \) and \( \beta(Y) \subseteq \beta(X) \). Now, if we add a vertex to \( Y \) from \( \beta(Y) \), we get a set \( X' \) such that \( |X'| \leq |X| \) and \( \beta(X') \subseteq \beta(X) \), and by choosing an appropriate superset of \( X' \), we get a contradiction to the minimality of \( \beta(X) \).