CHAPTER 3

Some equivalent problems

3.1. Pólya's permanent problem

Computing the permanent of a matrix seems to be of a different computational complexity from computing the determinant. While the determinant can be calculated using Gaussian elimination, no efficient algorithm for computing the permanent is known, and, in fact, none is believed to exist. More precisely, Valiant [**65**] has shown that computing the permanent is #P-complete even when restricted to 0-1 matrices.

It is therefore reasonable to ask if perhaps computing the permanent can be somehow reduced to computing the determinant of a related matrix. In particular, the following question was asked by Pólya [**49**] in 1913. If A is a 0-1 square matrix, does there exist a matrix B obtained from A by changing some of the 1's to -1's in such a way that the permanent of A equals the determinant of B? For the purpose of this paper let us say that B (when it exists) is a *Pólya matrix* for A.

Vazirani and Yannakakis [**66**] proved the following.

Theorem 3.1.1. *Let G be a bipartite graph, and let A be its bipartite adjacency matrix. Then A has a Pólya matrix if and only if G has a Pfaffian orientation.*

Exercise 3.1.2. Let G be a simple bipartite graph with bipartition $([n], [2n]-[n])$, let D be an orientation of G, and let $A = (a_{ij})$ be the $n \times n$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j+n) \in E(D) \\ -1 & \text{if } (j+n, i) \in E(D) \\ 0 & \text{otherwise} \end{cases}$$

Let M be a perfect matching in G, and let σ be a permutation of $[n]$ defined by saying that M consists of all edges of the form $\{i, \sigma(i)+n\}$. Then

$$\mathrm{sgn}_D(M) = \mathrm{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdot \ldots \cdot a_{n\sigma(n)}$$

Theorem 3.1.3. *Let G be a bipartite graph with bipartite adjacency matrix A, let D be an orientation of G, and let B be the directed bipartite adjacency matrix of D. Then $\mathrm{per}(A) = \det(B)$ if and only if D is a Pfaffian orientation of G.*

Corollary 3.1.4. *A bipartite graph is Pfaffian if and only if its bipartite adjacency matrix has a Pólya matrix.*

Exercise 3.1.5. Let A be an $n \times n$ matrix with nonnegative entries, and let G be the bipartite graph with vertices corresponding to rows and columns of A such that a vertex corresponding to row r is adjacent to a vertex corresponding to column c if and only if $a_{rc} \neq 0$. Then A has a Pólya matrix if and only if G has a Pfaffian orientation.

3.2. Even digraphs

Let us turn to directed graphs now. A digraph D is *even* if for every weight function $w : E(D) \to \{0,1\}$ there exists a cycle in D of even total weight. It was shown in [**55**] and is not difficult to see that testing evenness is polynomial-time equivalent to testing whether a digraph has an even directed cycle. (This is equivalent to Theorem 2.5.1 for bipartite graphs.) Let G be a bipartite graph with bipartition (A,B), and let M be a perfect matching in G. Let $D = D(G,M)$ be obtained from G by directing every edge from A to B, and contracting every edge of M. Little [**28**] has shown the following.

Lemma 3.2.1. *Let G be a bipartite graph, and let M be a perfect matching in G. Then G has a Pfaffian orientation if and only if $D(G,M)$ is not even.*

Exercise 3.2.2. Let G be a bipartite graph with bipartition (A,B), let M be a perfect matching in G, and let $D := D(G,M)$. Then the following conditions are equivalent:
(1) G is matching covered,
(2) $|N(X)| \geq |X| + 1$ for every nonempty proper subset X of A,
(3) D is strongly connected.

Theorem 3.2.3. *Let G be a bipartite graph with bipartition (A,B), let M be a perfect matching in G, let $D := D(G,M)$, and let k be an integer. Then the following conditions are equivalent:*
(1) *G is connected and k-extendable,*
(2) *for every set $X \subseteq A$ either $N(X) = B$ or $|N(X)| \geq |X| + k$,*
(3) *D is strongly k-connected.*

Exercise 3.2.4. Prove Theorem 3.2.3.

Exercise 3.2.5. Let G be a bipartite graph with bipartition (A,B) and let D be the orientation of G obtained by directing every edge from A to B. Then D is a Pfaffian orientation of G if and only if G has no central cycle of length divisible by four.

Exercise 3.2.6. Let D be an orientation of a bipartite graph with bipartition (A,B), let F be the set of all edges of D that are directed from B to A, and let C be a cycle in D. Then C is oddly oriented if and only if $|E(C) \cap F| + |E(C)|/2$ is odd.

Exercise 3.2.7. Let G be a bipartite graph with bipartition (A,B) and perfect matching M, let D be an orientation of G such that every edge of M is directed from A to B, let $D' := D(G,M)$, let C be an M-alternating cycle in G, let C' be the corresponding directed cycle of D', and let $w : E(D') \to \{0,1\}$ be defined by $w(e) = 1$ if e (regarded as an edge of G) is directed in D from A to B and $w(e) = 0$ otherwise. Then C is oddly oriented in D if and only if $w(C')$ is odd.

Theorem 3.2.8. *Let G be a bipartite graph with bipartition (A,B) and perfect matching M, and let $D := D(G,M)$. Then G has a Pfaffian orientation if and only if D is not even.*

3.3. An economics example

To be written

3.4. Sign-nonsingular matrices

We say that two $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ have the same *sign-pattern* if for all pairs of indices i, j the entries a_{ij} and b_{ij} have the same sign; that is, they are both strictly positive, or they are both strictly negative, or they are both zero. A square matrix A is *sign-nonsingular* if every real matrix with the same sign pattern is nonsingular.

In economic analysis one may not know the exact quantitative relationships between different variables, but there may be some qualitative information such as that one quantity rises if and only if another does. For instance, it is generally agreed that the supply of a particular commodity increases as the price increases, even though the exact dependence may vary. Thus we may want to deduce qualitative information about the solution to a linear system $A\mathbf{x} = \mathbf{b}$ from the knowledge of the sign-patterns of the matrix A and vector $\mathbf{b}$. That motivates the following definition. We say that the linear system $A\mathbf{x} = \mathbf{b}$ is *sign-solvable* if for every real matrix B with the same sign-pattern as A and every vector $\mathbf{c}$ with the same sign-pattern as $\mathbf{b}$ the system $B\mathbf{y} = \mathbf{c}$ has a unique solution $\mathbf{y}$, and its sign-pattern does not depend on the choice of B and $\mathbf{c}$. The study of sign-solvability was first proposed by Samuelson [**53**].

It follows from standard linear algebra that sign-solvability can be decided efficiently if and only if sign-nonsingularity can. But for square matrices the latter is equivalent to testing whether a given orientation of a bipartite graph is Pfaffian. To state the result, let D be a bipartite digraph with bipartition (X, Y). By Theorem 2.5.1 the following result implies that testing sign-solvability is polynomial-time equivalent to testing whether a bipartite graph is Pfaffian.

Theorem 3.4.1. *Let D be a directed bipartite graph, and let A be its directed bipartite adjacency matrix. Then A is sign-nonsingular if and only if D is a Pfaffian orientation of its underlying undirected graph.*

3.5. The polytope of even permutations

Our last problem is about the polytope of even permutation matrices. The convex hull of permutation matrices has been characterized by Birkhoff [**3**] as precisely the set of doubly stochastic matrices. It is an open problem to characterize the convex hull of even permutation matrices. More precisely, it is not known if there exists a polynomial-time algorithm to test whether a given $n \times n$ matrix belongs to this polytope. By a fundamental result of Grötschel, Lovász and Schrijver [**20**] this problem is solvable in polynomial time if there exists a polynomial-time algorithm for the optimization problem: Given a fixed $n \times n$ matrix M, find the maximum of $M \cdot X$ over all even permutation matrices X, where $\cdot$ denotes the dot product in $\mathbf{R}^{n^2}$ and both matrices are regarded as vectors of length n^2.

A special case of the above optimization problem when A is a 0-1 matrix and we want to determine if the maximum is n can be reformulated as follows. Let G be a bipartite graph with bipartition (A, B), and let D be the orientation of G defined by orienting every edge from A to B. The problem is: "Decide if G has a perfect matching M such that $\mathrm{sgn}_D(M) = 1$." By Theorem 2.5.1 this is polynomial-time equivalent to deciding whether a bipartite graph has a Pfaffian orientation.

Let $\vec{K}_n$ denote the complete directed multigraph with vertex-set $[n]$ and a loop at every vertex; that is, the edge-set of $\vec{K}_n$ is $[n] \times [n]$. By a *permutation digraph* on

$[n]$ we mean a directed submultigraph D of $\vec{K}_n$ such that $\deg_D^+(v) = \deg_D^-(v) = 1$ for every $v \in [n]$. Thus there is an obvious correspondence between permutation digraphs on $[n]$ and permutations of $[n]$. We say that a permutation digraph is *even* if the corresponding permutation is even.

Let $\mathcal{P}(n) \subseteq \mathbb{R}^{[n]\times[n]}$ denote the convex hull of characteristic vectors of edge-sets of permutation digraphs on $[n]$. A classical theorem of Birkhoff [**3**] characterizes $\mathcal{P}(n)$; we state it as an exercise.

Exercise 3.5.1. A vector $\mathbf{x} \in \mathbb{R}^{[n]\times[n]}$ belongs to $\mathcal{P}(n)$ if and only if

$$\sum_{i=1}^{n} x_{ik} = 1 \quad \text{for all } k = 1, 2, \ldots, n, \tag{1}$$

$$\sum_{j=1}^{n} x_{kj} = 1 \quad \text{for all } k = 1, 2, \ldots, n, \text{ and} \tag{2}$$

$$x_{ij} \geq 0 \quad \text{for all } i, j = 1, 2, \ldots, n. \tag{3}$$

Let $\mathcal{Q}(n)$ denote the convex hull of incidence vectors of edge-sets of even permutation digraphs. This polytope has been studied, but the complexity status of the following decision problem is not known.

3.5.2. THE EVEN PERMUTATION POLYTOPE MEMBERSHIP PROBLEM
Instance An integer $n \geq 1$ and a vector $\mathbf{x} \in \mathbb{Q}^{[n]\times[n]}$
Question Does $\mathbf{x} \in \mathcal{Q}(n)$?

Cunningham and Wang [**7**] pointed out that by a fundamental result of Grőtschel, Lovasz and Schrijver [**20**], the membership problem 3.5.2 is solvable in polynomial time if there is a polynomial-time algorithm for the following optimization problem:

3.5.3. OPTIMIZATION OVER THE EVEN PERMUTATION POLYTOPE
Instance A vector $\mathbf{c} \in \mathbb{R}^{[n]\times[n]}$
Objective Find the maximum of $\mathbf{c} \cdot \mathbf{x}$ over all $\mathbf{x} \in \mathcal{Q}(n)$.

The following is equivalent to the special case of Problem 3.5.3 when $\mathbf{c}$ is a 0-1 vector and we want to determine whether the maximum is n.

3.5.4. THE EVEN PERMUTATION DIGRAPH CONTAINMENT PROBLEM
Instance An integer n and a directed submultigraph D of $\vec{K}_n$
Question Does D contain an even permutation digraph on $[n]$

The next theorem implies that Problem 3.5.4 is polynomial-time equivalent to testing if a given orientation of a bipartite graph is Pfaffian.

Theorem 3.5.5. *Let D_0 be a directed submultigraph of $\vec{K}_n$, let D be the digraph with $V(G) = [2n]$ and $(2i-1, 2j) \in E(D)$ if and only if D_0 has a directed edge from i to j, and let G be the underlying undirected graph of D. Then every two permutation subdigraphs of D_0 have the same parity if and only if D is a Pfaffian orientation of G.*

The theorem follows immediately from the following exercise.

Exercise 3.5.6. Let D_0, D and G be as in Theorem 3.5.5. Let H be a permutation subdigraph of D_0 corresponding to a permutation σ, and let M be the perfect matching of G consisting of the edges of G that correspond to the edges of H. Then $\operatorname{sgn}(\sigma) = \operatorname{sgn}_{D_0}(M)$.

3.6. Minimally non-2-colorable hypergraphs

The next problem is about hypergraph coloring. A *hypergraph* is a finite set of distinct finite nonempty sets, called the *hyperedges* of $\mathcal{F}$. The set of vertices of $\mathcal{F}$, denoted by $V(\mathcal{F})$, is the union of all hyperedges of $\mathcal{F}$.

A partition (X_1, X_2) of $V(\mathcal{F})$ is a 2-*coloring* of $\mathcal{F}$ if every hyperedge of $\mathcal{F}$ intersects both X_1 and X_2. A hypergraph is 2-*colorable* if it has a 2-coloring.

Exercise 3.6.1. [**30**] Prove that deciding whether a hypergraph is 2-colorable is NP-hard. reference incorrect

We say that a hypergraph $\mathcal{F}$ is *minimally non*-2-*colorable* if $\mathcal{F}$ is not 2-colorable, but $\mathcal{F} - \{F\}$ is 2-colorable for every $F \in \mathcal{F}$. Following Seymour [**54**] we will call minimally non-2-colorable hypergraphs *condensers*.

Exercise 3.6.2. Prove that deciding whether a hypergraph is a condenser is NP-hard.

Exercise 3.6.3. Prove that the following hypergraphs are condensers:

(1) $\{\{1\}\}$
(2) $\{\{1,2\},\{2,3\},\{1,3\}\}$
(3) The set of lines of the Fano plane
(4) $\{\{1,2,\ldots,n\}\} \cup \{\{0,i\} : i = 1,2,\ldots,n\}$, where $n \geq 2$
(5) $\{\{1,2\},\{1,3\},\{1,4\},\{1,5,6\},\{2,3,4,5\},\{2,3,4,6\}\}$
(6) $\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$, where $n \geq 3$ is odd
(7) All k-element subsets of a $(2k-1)$-element set, where $k \geq 1$.

If $\mathcal{F}$ is a hypergraph and X is a set, then $\mathcal{F}|X$ denotes the hypergraph $\{F \cap X : F \in \mathcal{F}, F \cap X \neq \emptyset\}$.

Exercise 3.6.4. If $\mathcal{F}$ is a condenser and $X \subseteq V(\mathcal{F})$ is nonempty, then $\mathcal{F}|X$ is not 2-colorable.

Exercise 3.6.5. If $\mathcal{F}$ is a condenser and $f : V(\mathcal{F}) \to \mathbf{R}$ is such that $\sum_{v \in E} f(v) = 0$ for every $E \in \mathcal{F}$, then f is identically zero.

Exercise 3.6.6. If $\mathcal{F}$ is a condenser and $X \subseteq V(\mathcal{F})$ is such that no hyperedge intersects both X and its complement, then $X = \emptyset$ or $X = V(\mathcal{F})$.

Theorem 3.6.7. [**54**] *If $\mathcal{F}$ is a condenser and $X \subseteq V(\mathcal{F})$, then X intersects at least $|X|$ hyperedges of $\mathcal{F}$, with equality only if $X = \emptyset$ or $X = V(\mathcal{F})$.*

Proof. If $X = \emptyset$ the theorem holds. If $X = V(\mathcal{F})$ and X intersects fewer than $|X|$ hyperedges of $\mathcal{F}$, then $|\mathcal{F}| < |V(\mathcal{F})|$. Let M be the transpose of th incidence matrix of $\mathcal{F}$. Then M has fewer rows than columns, and hence there exists a nonzero vector $\mathbf{x}$ such that $M\mathbf{x} = \mathbf{0}$, contrary to Exercise 3.6.5.

It remains to show that every nonempty proper subset $X \subseteq V(\mathcal{F})$ intersects at least $|X| + 1$ hyperedges of $\mathcal{F}$. To this end suppose for a contradiction that X is a nonempty proper subset of $V(\mathcal{F})$ that intersects at most $|X|$ hyperedges of $\mathcal{F}$. By Exercise 3.6.6 there exists a hyperedge $E_0 \in \mathcal{F}$ that intersects both X and $V(\mathcal{F}) - X$. By the same linear algebra argument as above there exists a function $f : V(\mathcal{F}) \to \mathbf{R}$, not identically equal to zero, such that $\sum_{v \in E} f(v) = 0$ for every $E \in \mathcal{F} - \{E_0\}$ and $f(v) = 0$ for every $v \in V(\mathcal{F}) - X$. By Exercise 3.6.5 we have $\sum_{v \in E_0} f(v) \neq 0$, and so we may assume that $f(v_0) > 0$ for some $v_0 \in E_0$. Let $X^+ := \{v \in X : f(v) > 0\}$,

$X^- := \{v \in X : f(v) < 0\}$ and $\mathcal{F}' := \{F \in \mathcal{F} : \sum_{v \in F} f(v) = 0\}$. The hypergraph $\mathcal{F}'$ is 2-colorable by the minimality of $\mathcal{F}$. Let (Y_1, Y_2) be a 2-coloring of $\mathcal{F}'$; then $Y_1 \cup Y_2 \subseteq V(\mathcal{F}) - X^+ - X^-$, and by enlarging $Y_1 \cup Y_2$ if necessary we may assume that equality holds. From the symmetry we may assume that $E_0 \cap Y_2 \neq \emptyset$. It follows that $(X^+ \cup Y_1, X^- \cup Y_2)$ is a 2-coloring of $\mathcal{F}$, a contradiction. □

A cycle in a hypergraph $\mathcal{F}$ is a sequence $C = (v_1, F_1, v_2, F_2, \ldots, v_k, F_k)$ where
(1) $v_1, v_2, \ldots, v_k \in V(\mathcal{F})$ are pairwise distinct,
(2) $F_1, F_2, \ldots, F_k \in \mathcal{F}$ are pairwise distinct, and
(3) $v_i, v_{i+1} \in F_i$ for all $i = 1, 2, \ldots, k-1$, and $v_1, v_k \in F_k$. The *length* of C is k. We define $V(C) = \{v_1, v_2, \ldots, v_k\}$ and $E(C) = \{F_1, F_2, \ldots, F_k\}$. If $\mathcal{F}$ is a hypergraph we say that a set $X \subseteq V(\mathcal{F})$ can be *matched with* a set $\mathcal{F}' \subseteq \mathcal{F}$ if there exists a bijection $f : X \to \mathcal{F}'$ such that $v \in f(v)$ for every $v \in X$.

Theorem 3.6.8. [**54**] *A hypergraph $\mathcal{F}$ with $|\mathcal{F}| = |V(\mathcal{F})|$ is a condenser if and only if*
(1) every nonempty proper subset X of $V(\mathcal{F})$ intersects at least $|V(\mathcal{F})| + 1$ hyperedges of $\mathcal{F}$, and
(2) if C is a cycle $\mathcal{F}$ such that $V(\mathcal{F}) - V(C)$ may be matched with $\mathcal{F} - E(C)$, then the length of C is odd.

Proof. Let $\mathcal{F}$ be a hypergraph with $|\mathcal{F}| = |V(\mathcal{F})|$, and assume first that it is a condenser. Then $\mathcal{F}$ satisfies (1) by Theorem 3.6.7. To prove (2) let $C = (v_1, F_1, v_2, F_2, \ldots, v_k, F_k)$ be a cycle in $\mathcal{F}$ of even length and let $f : V(\mathcal{F}) - V(C) \to \mathcal{F} - E(C)$ be a bijection with $v \in f(v)$ for every $v \in V(\mathcal{F}) - V(C)$. Let us extend f to all of $V(\mathcal{F})$ by defining $f(v_i) = F_i$ for all $i = 1, 2, \ldots, k$. For $Z \subseteq V(\mathcal{F})$ we define $\mathcal{F}(Z)$ to be the hypergraph $\{Z \cap f(v) : v \in Z\}$. Then $\mathcal{F}(V(\mathcal{F})) = \mathcal{F}$ is not 2-colorable, but $\mathcal{F}(V(C))$ is 2-colorable, because $(\{v_1, v_3, \ldots, v_{k-1}\}, \{v_2, v_4, \ldots, v_k\})$ is a 2-coloring since k is even.

Let $Z \subseteq V(\mathcal{F})$ be maximal such that $\mathcal{F}(Z)$ is 2-colorable. Then Z is a nonempty proper subset of $V(\mathcal{F})$, and hence by Theorem 3.6.7 Z intersects at least $|Z| + 1$ hyperedges of $\mathcal{F}$. Thus there exists $v \in V(\mathcal{F}) - Z$ such that $Z \cap f(v) \neq \emptyset$. Let (Z_1, Z_2) be a 2-coloring of $\mathcal{F}(Z)$. From the symmetry we may assume that $f(v) \cap Z_1 \neq \emptyset$. But then $(Z_1, Z_1 \cup \{v\})$ is a 2-coloring of $\mathcal{F}(Z \cup \{v\})$, contradicting the maximality of Z. This concludes the proof that if $\mathcal{F}$ is a condenser, then it satisfies (1) and (2).

For the converse let $\mathcal{F}$ satisfy (1) and (2). We first show that $\mathcal{F}$ is not 2-colorable. To this end suppose that (Y_1, Y_2) is a 2-coloring of $\mathcal{F}$. By (1) and Hall's theorem there exists a bijection $f : V(\mathcal{F}) \to \mathcal{F}$ with $v \in f(v)$ for every $v \in V(\mathcal{F})$. Pick $v_1 \in Y_1$. Since (Y_1, Y_2) is 2-coloring, there exist $v_2 \in Y_2 \cap f(v_1)$, $v_3 \in Y_1 \cap f(v_2)$, $v_4 \in Y_2 \cap f(v_3)$, and so on. Since $\mathcal{F}$ is finite eventually we find a cycle of the form $C = (x_1, f(x_1), x_2, f(x_2), \ldots, x_k, f(x_k))$, where $x_i \in Y_1$ for odd i and $x_i \in Y_2$ for even i. Thus the cycle is even, and the restriction of f to $V(\mathcal{F}) - V(C)$ proves that $V(\mathcal{F}) - V(C)$ can be matched to $\mathcal{F} - E(C)$, contrary to (2). It remains to prove that if $\mathcal{F}' \subseteq \mathcal{F}$ is not 2-colorable, then $\mathcal{F}' = \mathcal{F}$. To this end we may assume that $\mathcal{F}' \subseteq \mathcal{F}$ is a condenser. By Theorem 3.6.7 we have $|\mathcal{F}'| \geq |V(\mathcal{F}')|$, and hence $|V(\mathcal{F}) - V(\mathcal{F}')| \geq |\mathcal{F} - \mathcal{F}'|$. By (1) applied to the set $X = V(\mathcal{F}) - V(\mathcal{F}')$ we deduce that either $X = \emptyset$ or $X = V(\mathcal{F})$. But $V(\mathcal{F}') \neq \emptyset$, and so $V(\mathcal{F}') = V(\mathcal{F})$. Consequently $|\mathcal{F}| \geq |\mathcal{F}'| \geq |V(\mathcal{F}')| = |V(\mathcal{F})| = |\mathcal{F}|$, and hence $\mathcal{F} = \mathcal{F}'$. Thus $\mathcal{F}$ is a condenser, as required. □

Theorem 3.6.9. *Let $\mathcal{F}$ be a hypergraph with $|\mathcal{F}| = |V(\mathcal{F})|$, let D be the directed graph with vertex-set $V(\mathcal{F}) \cup \mathcal{F}$ and edge-set $\{vF : v \in V(\mathcal{F}), F \in \mathcal{F}, v \in F\}$ and let G be the underlying undirected graph of D. Then $\mathcal{F}$ is a condenser if and only if G is matching covered and D is a Pfaffian orientation of G.*

Exercise 3.6.10. Prove Theorem 3.6.9

Exercise 3.6.11. Let $\mathcal{F}$ and D be as in Theorem 3.6.9, and let D' be obtained from D by contracting a perfect matching. Then $\mathcal{F}$ is a condenser if and only if D' is strongly connected and has no even cycle.

Let $\mathcal{F}$ be a hypergraph. A *transversal* of $\mathcal{F}$ is a set that intersects every hyperedge of $\mathcal{F}$. A *minimal transversal* is a transversal that includes no transversal as a proper subset. The *blocker* of $\mathcal{F}$, denoted by $b(\mathcal{F})$, is the hypergraph consisting of all minimal transversals of $\mathcal{F}$. A hypergraph $\mathcal{F}$ is a *clutter* if $F, F' \in \mathcal{F}$ and $F \subseteq F'$ implies $F = F'$.

Exercise 3.6.12. Prove that if $\mathcal{F}$ is a clutter, then $\mathcal{F} = b(b(\mathcal{F}))$.

A hypergraph $\mathcal{F}$ is *self-blocking* if $\mathcal{F} = b(\mathcal{F})$.

Exercise 3.6.13. Prove that a hypergraph $\mathcal{F}$ is self-blocking if and only if

(1) $\mathcal{F}$ is a clutter,
(2) every pair of edges of $\mathcal{F}$ intersect, and
(3) $\mathcal{F}$ is not 2-colorable.

Exercise 3.6.14. Characterize self-blocking hypergraphs $\mathcal{F}$ with $|\mathcal{F}| = |V(\mathcal{F})|$.

3.7. Hints for selected exercises

Hint for Exercise 3.2.2. The equivalence of (1) and (2) follows from Hall's theorem. The equivalence of (2) and (3) follows directly: if $|N(X)| = |X|$ for some nonempty proper subset X of A, then the partition $(X \cup N(X), V(G) - X - N(X))$ of $V(G)$ corresponds to a partition (U, V) of $V(D)$ that gives rise to a directed cut, and vice versa.

Hint for Exercise 3.2.4. Induction on k.

Hint for Exercise 3.2.7. By Exercise 3.2.6 C is oddly oriented if and only if $|E(C) \cap F| + |E(C)|/2$ is odd, where F is the set of edges of G directed in D from B to A. But we have (modulo 2)

$$\begin{aligned} |E(C) \cap F| + |E(C)|/2 &= |E(C) - F| + |E(C')| = \\ &= |E(C')| + |E(C') - F| + |E(C')| = w(C'). \end{aligned}$$

Hint for Exercise 3.6.4.[**54**] Suppose that (X_1, X_2) is a 2-coloring of $\mathcal{F}|X$. Let $\mathcal{G} = \{F \in \mathcal{F} : F \cap X = \emptyset\}$. By the minimality of $\mathcal{F}$ the hypergraph $\mathcal{G}$ is 2-colorable; let (Y_1, Y_2) be a 2-coloring of $\mathcal{G}$. Then $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a 2-coloring of $\mathcal{F}$, a contradiction.

Hint for Exercise 3.6.5.[**54**] Suppose f exists and is not identically zero, and let $X^+ := \{v \in V(\mathcal{F}) : f(v) > 0\}$, $X^- := \{v \in V(\mathcal{F}) : f(v) < 0\}$ and $X := X^+ \cup X^-$. Then (X^+, X^-) is a 2-coloring of $\mathcal{F}|X$, contrary to Exercise 3.6.4.

Hint for Exercise 3.6.6. Both $\mathcal{F}|X$ and $\mathcal{F}|(V(\mathcal{F}) - X)$ are 2-colorable.

